Maths 212, Homework #1
First four problems:
due Thursday, Oct. 20

1. Let $A = \mathbb{Q} \cap (0, \pi)$ be the set of all rational numbers between 0 and π. Find the least upper bound of A and prove that your answer is correct.

2. Let A be a bounded set of real numbers and let $B \subset A$. Show that B is also bounded and that $\sup B \leq \sup A$. Establish a similar relation between $\inf A$ and $\inf B$.

3. Define a sequence $\{s_n\}$ by setting $s_1 = 1$ and then
$$s_{n+1} = \frac{2s_n + 2}{s_n + 2}$$
for each $n \geq 1$. Show that this sequence is convergent and find its limit.

4. Let $\{s_n\}$ be a convergent sequence and denote its limit by L. Given some $L' < L$, show that there exists an integer N such that $s_n > L'$ for all $n \geq N$.

5. (Squeeze law) Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences with $a_n \leq b_n \leq c_n$ for each integer n. Assuming that $a_n \to L$ and $c_n \to L$, show that $b_n \to L$ as well.

6. Let $\{s_n\}$ be a convergent sequence. Intuitively, s_{n+1} and s_n should be approaching the exact same limit. Verify this rigorously by showing that $s_{n+1} - s_n \to 0$.

7. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \left\{ \begin{array}{ll} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$
Determine all points at which f is continuous.