UNIVERSITY OF DUBLIN

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
JS Two Subject Moderatorship
SS Two Subject Moderatorship

Trinity Term 1999

COURSE 212

Tuesday, June 1 Luce Hall 14.00 — 17.00

Dr. D. R. Wilkins

Credit will be given for the best 7 questions answered. Logarithmic tables will be available in the examination hall.

1. Let a and b be real numbers satisfying $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function on the closed bounded interval $[a,b]$.

   (a) Prove that there exists a constant M with the property that $|f(x)| \leq M$ for all $x \in [a,b]$.

   (b) Prove that there exist real numbers u and v in the interval $[a,b]$ with the property that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a,b]$.

   (c) Suppose that $f(x) < g(x)$ for all $x \in [a,b]$, where $g\colon [a,b] \to \mathbb{R}$ is a continuous function on $[a,b]$ and $g(x) > 0$ for all $x \in [a,b]$. Prove that there exists a real number θ satisfying $0 < \theta < 1$ (where θ is independent of x) such that $f(x) \leq \theta g(x)$ for all $x \in [a,b]$.

2. Determine which of the following subsets of $\mathbb{R}^2$ are open in $\mathbb{R}^2$ and which are closed in $\mathbb{R}^2$, giving reasons for your answers:—

   (i) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + 2x \geq 3 \text{ or } x^2 + y^2 - 2x \geq 3\}$,

   (ii) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + 2x < 3 \text{ and } x^2 + y^2 - 2x \geq 3\}$,

   (ii) $\{(x,y) \in \mathbb{R}^2 : y^2 = x(x^2 - 1) \text{ and } x > 0\}$.

3. (a) What is a *metric space*?
   (b) What is an *open set* in a metric space? What is a *closed set* in a metric space?
   (c) Prove that any union of open sets in a metric space is an open set. Prove also that any finite intersection of open sets in a metric space is an open set.
   (d) Let X be a metric space with distance function d, and let u and v be points of X. Let $W = \{x \in X : d(x,u) < d(x,v)\}$. Prove that W is an open set in X.

4. (a) Let X and Y be metric spaces. Define is meant by saying that a function $f: X \to Y$ is *continuous*. [Your definition should be expressed in terms of the distance functions on the metric spaces X and Y, and should not make reference to open or closed sets.]
   (b) Let X and Y be metric spaces, and let $f: X \to Y$ be a function from X to Y. Prove that the function $f: X \to Y$ is continuous if and only if the preimage $f^{-1}(V)$ of every open set V in Y is an open set in X.
   (c) Let X and Y be metric spaces, and let $f: X \to Y$ be a continuous function from X to Y. Let G be a closed set in Y. Explain why the preimage $f^{-1}(G)$ of G is a closed set in X.

5. (a) What is a *topological space*?
   (b) Let $X_1, X_2, \ldots, X_n$ be topological spaces. Give the definition of the *product topology* on the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of $X_1, X_2, \ldots, X_n$, and prove that the collection of open sets in $X_1 \times X_2 \times \cdots \times X_n$ satisfies the axioms in the definition of a topological space.
   (c) Prove that the product topology on $\mathbb{R}^n$ (obtained on regarding $\mathbb{R}^n$ as the Cartesian product of n copies of the real line $\mathbb{R}$) is the same as the usual topology on $\mathbb{R}^n$ generated by the Euclidean distance function on $\mathbb{R}^n$.

6. (a) What is a *compact* topological space?
   (b) Let X and Y be topological spaces, let $f: X \to Y$ be a continuous function from X to Y, and let K be a compact subset of X. Prove that $f(K)$ is a compact subset of Y.
   (c) Explain why any closed subset of a compact topological space is compact.
   (d) Prove that any compact subset of a metric space is closed.
   (e) Prove that a subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded. [You may use without proof the result that any finite Cartesian product of compact topological spaces is compact.]

7. (a) What is a *connected* topological space?

   (b) Prove that a topological space X is connected if and only if every continuous function $f: X \to \mathbb{Z}$ from X to the set of integers is constant.

   (c) Let $f: X \to Y$ be a continuous function between topological spaces X and Y, and let A be a connected subset of X. Prove that $f(A)$ is a connected subset of Y.

   (d) What are the connected components of $\{(x, y, z) \in \mathbb{R}^3 : z^2 - x^2 - y^2 = 1\}$? [Justify your answer.]

8. **(a)** What is a *norm* on a real or complex vector space? What is meant by saying that two norms on a real or complex vector space are *equivalent*?

   (b) Let $\|.\|$ be a norm on $\mathbb{R}^n$. Prove that the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous with respect to the usual topology on $\mathbb{R}^n$.

   (c) Prove that any two norms on $\mathbb{R}^n$ are equivalent, and induce the usual topology on $\mathbb{R}^n$. [You may use without proof the result that if two norms on a real or complex vector space are both equivalent to some third norm then they are equivalent to each other.]

9. **(a)** What is meant by saying that an open set U in $\mathbb{C} \setminus \{0\}$ is *evenly covered* by the exponential map $\exp: \mathbb{C} \to \mathbb{C} \setminus \{0\}$?

   (b) Let $\gamma: [0, 1] \to \mathbb{C} \setminus \{0\}$ be a continuous path in $\mathbb{C} \setminus \{0\}$, and let z be a complex number satisfying $\exp(z) = \gamma(0)$. Proof that there exists a unique continuous path $\tilde{\gamma}: [0, 1] \to \mathbb{C}$ such that $\tilde{\gamma}(0) = z$ and $\exp \circ \tilde{\gamma} = \gamma$. [This is the *Path-Lifting Theorem* satisfied by the exponential map.]

10. **(a)** Let $\gamma: [0, 1] \to \mathbb{C}$ be a closed curve in the complex plane (where $\gamma(0) = \gamma(1)$), and let w be a complex number that does not lie on the curve. Give the definition of the *winding number* $n(\gamma, w)$ of the closed curve γ about w.

    (b) Let w be a complex number and, for each $\tau \in [0, 1]$, let $\gamma_\tau: [0, 1] \to \mathbb{C}$ be a closed curve in $\mathbb{C}$ which does not pass through w. Suppose that the map sending $(t, \tau) \in [0, 1] \times [0, 1]$ to $\gamma_\tau(t)$ is a continuous map from $[0, 1] \times [0, 1]$ to $\mathbb{C}$. Using the Monodromy Theorem, or otherwise, prove that $n(\gamma_0, w) = n(\gamma_1, w)$.

    (c) State and prove the *Fundamental Theorem of Algebra*