TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
JS Two Subject Moderatorship
SS Two Subject Moderatorship

Trinity Term 1998

COURSE 212

Wednesday, May 20 Room 4050B 09.30 — 12.30

Dr. D.R. Wilkins

Credit will be given for the best 7 questions answered. Logarithmic tables will be available in the examination hall

1. Determine which of the following subsets of $\mathbb{R}^3$ are open in $\mathbb{R}^3$ and which are closed in $\mathbb{R}^3$, giving reasons for your answers: -

   (i) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 4 \text{ and } y \leq 1\}$,

   (ii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \geq 4 \text{ and } y < 1\}$,

   (iii) $\{(x, y, z) \in \mathbb{R}^3 : x > 0 \text{ and } x^2 - y^2 \geq 1\}$.

2. Let S^2 denote the unit sphere $\{(x, y, x) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$, and let $\mathbf{n} = (0, 0, 1)$. For each point $\mathbf{x}$ of $S^2 \setminus \{\mathbf{n}\}$, let $\varphi(\mathbf{x})$ be the point where the line through $\mathbf{n}$ and $\mathbf{x}$ intersects the plane

$$H = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$$

   Prove that $\varphi : S^2 \setminus \{\mathbf{n}\} \to H$ is a homeomorphism.

3. (a) What is a *metric space*?

   (b) What is an *open set* in a metric space? What is a *closed set* in a metric space?

   (c) What is meant by saying that a sequence $x_1, x_2, x_3, \ldots$ of points in a metric space X *converges* to some point p of X?

   (d) Let F be a closed set in a metric space X. Suppose that some sequence $x_1, x_2, x_3, \ldots$ of points of F converges to some point p of X. Prove that p belongs to F.

   (e) Let $f : X \to \mathbb{R}$ be a continuous function on a metric space X, and let $x_1, x_2, x_3, \ldots$ be a sequence of points of X that converges to some point p of X. Let c be a real number. Suppose that $f(x_j) \leq c$ for all natural numbers j. Explain why $f(p) \leq c$.

4. (a) What is a *topological space*?

   (b) A subset N of a topological space is said to be a *neighbourhood* of a point of x of X if there exists some open set U such that $x \in U$ and $U \subset N$. Prove that a subset V of a topological space X is open in X if and only if V is a neighbourhood of each point of V.

   (c) Let A be a subset of a topological space X. The *interior* A^o of A is the union of all open sets contained in A. What are the interiors of the following subsets of $\mathbb{R}^2$:

   (i) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$?

   (ii) $\{(x, y) \in \mathbb{R}^2 : y = 7\}$?

   (d) Let A be a subset of a topological space X and let $x_1, x_2, x_3, \ldots$ be an infinite sequence of points of X that converges to some point p of the interior of A. Explain why there exists some positive integer N such that $x_j \in A$ whenever $j \geq N$.

5. (a) What is the product topology on a Cartesian product of topological spaces $X_1, X_2, \ldots, X_n$.

   (b) Prove that a function $f : Z \to X_1 \times X_2 \times \cdots \times X_n$ from a topological space Z to the Cartesian product of topological spaces $X_1, X_2, \ldots, X_n$ is continuous if and only if the components $f_1, f_2, \ldots, f_n$ of f are continuous. (Here

$$f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$$

   for all $z \in Z$.)

   (c) The topological space $\mathbf{R}^n$ can be regarded as the Cartesian product of n copies of the space $\mathbf{R}$. Prove that the product topology on $\mathbb{R}^n$ coincides with the usual topology generated by the Euclidean norm on $\mathbb{R}^n$

6. (a) What is a *compact* topological space?

   (b) Let $f : X \to \mathbb{R}$ be a continuous real-vauled function on a compact topological space X. Prove that f is bounded by the above and below on X. Prove also that there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.

   (c) Let A be a compact subset of a metric space X. Prove that A is closed in X.

7. (a) What is meant by saying that a topological space is *connected*? What is meant by saying that a topological space is *path-connected*?

   (b) Prove that a topological space is connected if and only if every continuous function from X to the set $\mathbb{Z}$ of integers is constant.

   (c) Prove that every path-connected topological space is connected.

   (d) What are the connected components of

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 2x\}?$$

   [Justify your answer.]

8. (a) What is a *normed vector space* (over the field of real or complex numbers). What is a *Banach space*?

   (b) Let X be a Banach space, and let $x_1, x_2, x_3, \ldots$ be elements of X. Suppose that $\sum_{n=1}^{+\infty} \|x_n\|$ is convergent. Prove that $\sum_{n=1}^{\infty} x_n$ is convergent, and

$$\left\| \sum_{n=1}^{+\infty} x_n \right\| \leq \sum_{n=1}^{+\infty} \|x_n\|$$

9. State and prove the *Contraction Mapping Theorem*.

10. Write an account of the theory of *winding numbers* of closed curves in the complex plane, selecting aspects of the theory that you consider to be important.