UNIVERSITY OF DUBLIN

XMA2121

TRINITY COLLEGE

Faculty of Science

school of mathematics

SF Mathematics **Trinity Term 2006**
JS Two Subject Moderatorship
SS Two Subject Moderatorship

Course 212

Saturday, June 10 GMB 9:30 – 12:30

Dr. P. Karageorgis

Credit will be given for the best 6 questions answered.

1. Let X be an arbitrary metric space and let $A \subset X$ be a subset.

(a) When is A said to be open in X? When is a point $x \in X$ a limit point of A?

(b) Show that every metric space is necessarily Hausdorff.

(c) Show that the collection of all open subsets of X forms a topology on X.

2.

(a) What is a partition of a topological space X? When is X said to be connected?

(b) Show that every path connected topological space must also be connected.

(c) Show that the unit interval $[0, 1]$ is a connected subset of $\mathbb{R}$ with its usual topology.

3. Let X be an arbitrary topological space and let $A \subset X$ be a subset.

(a) How is the closure of A defined? the interior of A? the boundary?

(b) Show that the interior does not intersect the boundary, i.e., that $\operatorname{Int} A \cap \operatorname{Bd} A = \varnothing$.

(c) Find the closure, the interior and the boundary of each of the following subsets of $\mathbb{R}$:

$$A_1 = [0, 1) \cup \{2\}, \qquad A_2 = \mathbb{Z}, \qquad A_3 = \mathbb{Q}, \qquad A_4 = (0, 1) \cap \mathbb{Q}.$$

Assume $\mathbb{R}$ is equipped with its usual topology here; you need not justify your answers.

4.

(a) When is a topological space compact? sequentially compact? totally bounded?

(b) Show that every compact metric space is limit point compact.

(c) Show that every compact metric space is both totally bounded and complete.

5.

(a) Show that the function $f(x) = 1/x$ is uniformly continuous on $(1, 2)$.

(b) Show that the sequence $f_n(x) = x^n(1 - x)$ converges uniformly on $[0, 1]$.

(c) Show that every continuous function $f\colon [0, 1] \to \mathbb{R}$ is uniformly continuous.

(d) Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions such that $f_n \to f$ uniformly. Given a sequence of points $x_n \in \mathbb{R}$ with $x_n \to 0$, show that $f_n(x_n) \to f(0)$ as well.

6. Let $\gamma_0, \gamma_1\colon [0, 1] \to X$ be paths in a topological space X.

(a) When is γ_0 said to be homotopic to γ_1? When is X said to be contractible?

(b) Show that a retract of a contractible space must also be contractible.

(c) Show that the unit circle S^1 is not a retract of the unit disc B^2.

(d) Show that every contractible space is path connected.

7. Let $p\colon Y \to X$ be a function between topological spaces.

(a) When is p said to be a covering map? a quotient map?

(b) Give an example of a covering map and one of a quotient map. You need not check that your maps satisfy the properties of part (a).

(c) Show that the composition of two quotient maps is a quotient map itself.

(d) Show that every covering space of a Hausdorff space must also be Hausdorff.

8.

(a) Find the fundamental group of the unit sphere S^2 with one point removed.

(b) Find the fundamental group of the complement of the z-axis in $\mathbb{R}^3$.

(c) Compute the simplicial homology groups of the torus.