UNIVERSITY OF DUBLIN

XMA2121

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
JS Two Subject Moderatorship
SS Two Subject Moderatorship

Trinity Term 2005

COURSE 212

Saturday, June 11 GMB 09.30 — 12.30

Dr. D. Zaitsev

Credit will be given for the best SIX questions answered

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. *(i)* Define a metric.

(ii) Give an example of a metric on $\mathbb{R}$ different from the standard one.

(iii) Let d_1 and d_2 be two metrics on the same set X, is it always true that $d := d_1 + 2d_2$ is also a metric? Justify your answer.

2. Find the interior and the boundary of the following subsets of $\mathbb{R}^2$:

(i) $\{(x, y) : x + y < 1\}$;

(ii) $\{(x, y) : xy = -1\}$;

(iii) $\{(x, y) : x \in \mathbb{Q}, y > 0\}$;

(iv) $\{(x, y) : x - y \in \mathbb{R} \setminus \mathbb{Q}\}$;

3. Prove or disprove:

(i) Any map from a discrete metric space into a general metric space is continuous.

(ii) Any map from a metric space into a discrete metric space is continuous.

(iii) Any set in a discrete metric space is bounded.

(iv) Any set in a discrete metric space is open.

4.

(i) Define a complete metric space and give an example of metric space which is not complete.

(ii) State and prove Cantor's intersection theorem. Is it still true without completeness? Justify your answer.

5.

(i) Define a topological space.

(ii) What is discrete and what is indiscrete topological space?

(iii) Give an example of a topology on the set $\{1, 2, 3\}$ which is neither discrete nor indiscrete. Justify your answer.

6. *(i)* Define a Hausdorff topological space.

(ii) Prove that in a Hausdorff topological space, the limit of a sequence of points is unique.

(iii) What is the relation between compactness and closedness for subsets in a Hausdorff topological space?

7. Let X and Y be two metric spaces, $B \subset Y$ a subset and $f: X \to Y$ a continuous map. Prove or disprove:

(i) B is closed in $Y \Longrightarrow f^{-1}(B)$ is closed in X;

(ii) B is compact $\Longrightarrow f^{-1}(B) \subset X$ is compact;

(iii) B and X are compact $\Longrightarrow f^{-1}(B)$ is compact;

(iv) B is connected $\Longrightarrow f^{-1}(B)$ is connected;

8.

(i) Define a normed vector space and a bounded linear operator between two normed spaces.

(ii) Show that a linear operator is continuous if and only if it is bounded.

(iii) Define the norm of a bounded operator. Give an example of an operator on $\mathbb{R}^2$ with norm 1.

9.

(i) Define the fundamental group of a topological space and prove that it is indeed a group.

(ii) Give an example of a topological space with a nontrivial fundamental group. Justify your answer.