UNIVERSITY OF DUBLIN

MA2121

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
JS Two Subject Moderatorship
SS Two Subject Moderatorship

Trinity Term 2004

COURSE MATHS: 212

Monday, May 17 RDS 14.00 — 17.00

Dr. D. Zaitsev

Credit will be given for the best SIX questions answered

1. (a) Define a metric space. Give an example of a metric space (X, d) and of a pair of a set X and a function $d: X \times X \to \mathbb{R}$ such that (X, d) is not a metric space.

   (b) Define descrete metric space.

   (c) Define a metric d on the set of all continuous functions $X = C([a, b])$ on an interval $[a, b] \subset \mathbb{R}$ and show that (X, d) is a metric space.

2. Which of the following subsets of $\mathbb{R}^2$ are open and which are closed in $\mathbb{R}^2$? Justify your answer.

   (a) $\{(x, y) \in \mathbb{R} : 1 < x^2 + y < 2\}$;

   (b) $\{(x, y) \in \mathbb{R} : x < 0,\ xy = 1\}$;

   (c) $\{(x, y) \in \mathbb{R} : x < 0,\ y \geq 0\}$.

   (d) $\{(x, y) \in \mathbb{R} : x^2 + y^2 \in \mathbb{Q}\}$.

3. (a) Define a continuous function between metric spaces (in terms of metrics). Show that any function between discrete metric spaces is continuous.

   (b) Given a continuous function $f: M \to \mathbb{R}$, show that its square $f^2: M \to \mathbb{R}$ is also continuous.

4. (a) Define a topological space and show that any metric space carries a natural (induced) topology.

(b) Define convergent sequences in a metric space using the metric and in a topological space using the topology. Show that in a metric space both definitions are equivalent, i.e. a sequence converges to a limit point in the induced topology if and only if it converges to the same point with respect to the given metric.

(c) Define a Hausdorff topological space and show that the induced topology of a metric space is always Hausdorff.

5. (a) Which of the following subspaces of $\mathbb{R}^n$ (for suitable n) are complete? Justify your answer.

   (i) $\{1/n^2 : n \in \mathbb{N}\}$;

   (ii) $\mathbb{Q} \times \mathbb{R}$;

   (iii) $\{(x, y) \in \mathbb{R}^2 : y \geq \frac{1}{1+x^2}\}$.

   (b) State Banach's contraction principle.

6. Consider two topological spaces X, Y, and let f be a continuous map $f\colon X \to Y$. Prove or disprove the following:

   (i) $A \subseteq X$ closed $\Longrightarrow f(A) \subseteq Y$ closed;

   (ii) X compact $\Longrightarrow f(X)$ compact;

   (iii) X connected $\Longrightarrow f(X)$ connected;

   (iv) X, Y metric spaces; X complete $\Longrightarrow f(X)$ complete.

7. Prove or disprove:

   (a) any continuous function on a compact topological space has Intermediate Value Property;

   (b) any continuous function on a compact topological space is bounded;

   (c) any continuous function on a connected topological space has Intermediate Value Property;

   (d) any continuous function on a connected topological space is bounded.

8. (a) What is a Banach space?

   (b) Define equivalent norms and give an example of two different equivalent norms on a vector space.

   (c) Show that a linear operator between normed spaces is bounded if and only if it continuous.

9. (a) When two closed curves in a topological space are called homotopic?

   (b) For a closed curve in the complex plane $\mathbb{C}$ define the winding number and show that homotopic curves have the same winding number.

   (c) Give an example of two connected open sets in $\mathbb{R}^2$ that are not homeomorphic. Justify your answer.