UNIVERSITY OF DUBLIN

MA2121

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
SF Theoretical Physics
SF TSM

Trinity Term 2003

COURSE 212

Thursday, May 22 RDS–Simmonscourt 14.00 — 17.00

Dr J. Bennett

Credit will be given for the best 6 questions answered.

1. (i) State the Completeness Axiom for the real numbers.

   (ii) Prove that a bounded monotone sequence of real numbers is convergent.

   (iii) Show that any sequence of real numbers $\{a_n\}$ has a monotone subsequence.

   (iv) Deduce the Bolzano–Weierstrass Theorem for bounded sequences of real numbers.

2. Let (X, d_X) and (Y, d_Y) be metric spaces.

   (i) Define what it means for a function $f : X \to Y$ to be continuous. [The definition you give should make no explicit reference to open sets.]

   (ii) Prove that if $f : X \to Y$ is continuous then $f^{-1}(U)$ is open in X whenever U is open in Y.

   (iii) Determine which of the following subsets of $\mathbb{R}^3$ are open and which are closed, giving reasons for your answers.

   (a) $\{(x, y, z) \in \mathbb{R}^3 : xyz \geq 1\}$

   (b) $\{(x, y, z) \in \mathbb{R}^3 : x^2 < z < y^2\}$

3. (i) Let X be a metric space with metric d. What does it mean for a subset U of X to be open.

   (ii) Prove that the union of an arbitrary collection of open sets in a metric space is an open set. Is this true if the word "open" is replaced with "closed"? Justify your answer.

   (iii) Prove that for each $x \in X$ and $\epsilon > 0$,

$$\{x' \in X : d(x, x') < \epsilon\}$$

   is open in X.

   (iv) Prove that a subset U of a metric space is open *if and only if* it can be expressed as a union of open balls.

4. (i) What is meant by a continuous function between topological spaces?

   (ii) What is a Hausdorff space?

   (iii) Suppose that X is a Hausdorff space and $f : X \to X$ is continuous. If $x \in X$ is such that $f(x) \neq x$, show that there exists an open set U_x in X, containing x, for which $f(y) \neq y$ for all $y \in U_x$.

   (iv) Show that the set of fixed points of f

$$\{x \in X : f(x) = x\}$$

   is closed in X.

   (v) Give an example of a (discontinuous) function $f : \mathbb{R} \to \mathbb{R}$ for which $\{x \in \mathbb{R} : f(x) = x\}$ is *not* closed.

5. (i) What does it mean for a topological space to be *compact*?

   (ii) Let X and Y be topological spaces with X compact. Prove that if $f : X \to Y$ is a continuous function then $f(X)$ is a compact subset of Y.

   (iii) State the Heine–Borel Theorem for subsets of $\mathbb{R}^n$.

   (iv) Prove that if a continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ is such that $f(\omega) \neq 0$ for all $\omega \in \mathbb{S}^{n-1}$, then there exists $\epsilon > 0$ such that $|f(\omega)| \geq \epsilon$ for all $\omega \in \mathbb{S}^{n-1}$.

   [Here $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ denotes the unit sphere in $\mathbb{R}^n$.]

6. (i) What does it mean for a topological space to be *connected*? Give *three* further equivalent statements.
   (ii) Let X and Y be topological spaces with X connected. Prove that if $f : X \to Y$ is continuous then $f(X)$ is a connected subset of Y.
   (iii) Prove that $\mathbb{R}$ is connected. [You may use any standard results for functions of a real variable provided you state them clearly.]
   (iv) Show that
   $$\{(x + x^2, x - x^4) : x \in \mathbb{R}\}$$
   is a connected subset of $\mathbb{R}^2$.

7. (i) State and prove the Contraction Mapping Theorem.
   (ii) Let $f_0 \in C([0,1])$ be fixed. Prove that there is a unique $f \in C([0,1])$ satisfying the integral equation
   $$f(t) = f_0(t) + \int_0^{t/2} f(s)ds$$
   for all $t \in [0,1]$.

8. (i) What is a *norm* on a real vector space X?
   (ii) State the Riesz Lemma for normed vector spaces.
   (iii) Let X be a normed vector space over $\mathbb{R}$. Suppose that Ω is a subset of X such that
   $$\{x \in X : \|x\| = 1\} \subset \Omega.$$
   Use the Riesz Lemma to prove that if Ω is compact then X must be finite dimensional.

9. Let X and Y be topological spaces and let $x_0 \in X$.
   (i) Define the *fundamental group* $\pi_1(X, x_0)$. [Your definition should include a description of the group law.]
   (ii) Show that if a continuous function $h : X \to X$ is homotopic to the identity I_X on X, then the induced homomorphism
   $$h_* : \pi_1(X, x_0) \to \pi_1(X, h(x_0))$$
   is an isomorphism.
   (iii) Show that if there are continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to I_X, then $\pi_1(X, x_0)$ is isomorphic to a subgroup of $\pi_1(Y, f(x_0))$.
   (iv) Explain why $\mathbb{S}^1$ is not a retract of D^2.