UNIVERSITY OF DUBLIN

MA2121

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
JS Two Subject Moderatorship
SS Two Subject Moderatorship

Trinity Term 2001

COURSE 212

Wednesday, May 30 RDS-Simmonscourt 1 14.00 — 17.00

Dr. D. R. Wilkins

Credit will be given for the best 6 questions answered. Logarithmic tables will be available in the examination hall.

1. State and prove the *Intermediate Value Theorem.*

2. Determine which of the following subsets of $\mathbb{R}^2$ are open in $\mathbb{R}^2$ and which are closed in $\mathbb{R}^2$, giving reasons for your answers:—

   (i) $\{(x,y) \in \mathbb{R}^2 : -1 < x < 1 \text{ and } -1 < y < 1\}$;

   (ii) $\{(x,y) \in \mathbb{R}^2 : x > 0,\ y > 0 \text{ and } xy = 1\}$;

   (iii) $\{(x,y) \in \mathbb{R}^2 : -1 < x < 1 \text{ and } -1 \leq y \leq 1\}$.

3. (a) What is a *metric space*?

   (b) What is an *open set* in a metric space? What is a *closed set* in a metric space?

   (c) What is meant by saying that a sequence $x_1, x_2, x_3, \ldots$ of points in a metric space X *converges* to some point p of X.

   (d) Prove that a sequence $x_1, x_2, x_3, \ldots$ of points in a metric space X converges to a point p of X if and only if, given any open set U in X such that $p \in U$, there exists some positive integer N such that $x_j \in U$ whenever $j \geq N$.

   (e) Let F be a closed set in a metric space X. Suppose that a sequence $x_1, x_2, x_3, \ldots$ of points of F converges to a point p of X. Prove that p belongs to F.

   (f) Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ be infinite sequences of real numbers. Suppose that $x_j^2 + y_j^2 \leq 1$ for all positive integers j, and that $x_j \to a$ and $y_j \to b$ as $j \to \infty$, where a and b are real numbers. Explain why $a^2 + b^2 \leq 1$.

4. (a) Let X and Y be metric spaces. What is meant by saying that a function $f\colon X \to Y$ is *continuous*? [Your definition should be expressed in terms of the distance functions on the metric spaces X and Y, and should not make reference to open or closed sets.]

(b) Let X and Y be metric spaces, and let $f\colon X \to Y$ be a function from X to Y. Prove that the function $f\colon X \to Y$ is continuous if and only if the preimage $f^{-1}(V)$ of every open set V in Y is an open set in X.

(c) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function from $\mathbb{R}$ to itself. Explain why the set

$$\{(x,y) \in \mathbb{R}^2 : x^2y^2 < f(x)\}$$

is an open set in $\mathbb{R}^2$.

5. (a) What is a *topological space*?

(b) Let X and Y be topological spaces. What is meant by saying that a function $q\colon X \to Y$ from X to Y is an *identification map*?

(c) Let X be a topological space, let Y be a set, and let $q\colon X \to Y$ be a surjective function. Prove that there is a unique topology on Y for which the function $q\colon X \to Y$ is an identification map.

(d) Let X and Y be topological spaces, and let $q\colon X \to Y$ be an identification map. Let Z be a topological space, and let $f\colon Y \to Z$ be a function from Y to Z. Prove that the function $f\colon Y \to Z$ is continuous if and only if the composition function $f \circ q\colon X \to Z$ is continuous.

(e) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x - 1 & \text{if } x \geq 1; \\ 0 & \text{if } -1 \leq x \leq 1; \\ x + 1 & \text{if } x \leq -1. \end{cases}$$

Is the function $f\colon \mathbb{R} \to \mathbb{R}$ an identification map? [Justify your answer.]

6. (a) What is meant by saying that a topological space is *compact*?

(b) What is a *Hausdorff* topological space?

(c) Let X be a Hausdorff topological space, and let K be a compact subset of X. Let x be a point of $X \setminus K$. Prove that there exist open sets V and W in X such that $x \in V$, $K \subset W$ and $V \cap W = \emptyset$.

(d) Explain why every compact subset of a Hausdorff topological space is closed.

7. (a) What is meant by saying that a topological space is *connected*? What is meant by saying that a topological space is *path-connected*?

(b) Prove that a topological space X is connected if and only if every continuous function $f\colon X \to \mathbb{Z}$ from X to the set of integers is constant.

(c) Prove that every path-connected topological space is connected.

(d) What are the connected components of $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 0\}$? [Justify your answer.]

8. (a) State and prove the *Contraction Mapping Theorem.*

   (b) Let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be a function from $\mathbb{R}^3$ to itself. Suppose that $f(\mathbf{0}) = \mathbf{0}$ and

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq |\mathbf{x} - \mathbf{y}|$$

   for all points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^3$. Prove that, given any point $\mathbf{z}$ of $\mathbb{R}^3$ satisfying $|\mathbf{z}| \leq 1$, there exists a point $\mathbf{x}$ of $\mathbb{R}^3$ such that $|\mathbf{x}| \leq 1$ and $\mathbf{z} = 2\mathbf{x} - f(\mathbf{x})$.

9. (a) Let $\gamma: [0, 1] \to \mathbb{C}$ be a closed curve in the complex plane (where $\gamma(0) = \gamma(1)$), and let w be a complex number that does not lie on the curve. Give the definition of the *winding number* $n(\gamma, w)$ of the closed curve γ about w.

   (b) Let w be a complex number and, for each $\tau \in [0, 1]$, let $\gamma_\tau: [0, 1] \to \mathbb{C}$ be a closed curve in $\mathbb{C}$ which does not pass through w. Suppose that the map sending $(t, \tau) \in [0, 1] \times [0, 1]$ to $\gamma_\tau(t)$ is a continuous map from $[0, 1] \times [0, 1]$ to $\mathbb{C}$. Using the Monodromy Theorem, or otherwise, prove that $n(\gamma_0, w) = n(\gamma_1, w)$.

   (c) State and prove the *Fundamental Theorem of Algebra.*