UNIVERSITY OF DUBLIN

MA2121

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Mathematics
JS Two Subject Moderatorship
SS Two Subject Moderatorship

Trinity Term 2000

COURSE 212

Wednesday, May 31 Sports Hall 14.00 — 17.00

Dr. D. R. Wilkins

Credit will be given for the best 6 questions answered. Logarithmic tables will be available in the examination hall.

1. State and prove the *Bolzano-Weierstrass Theorem.*

2. Determine which of the following subsets of $\mathbb{R}^2$ are open in $\mathbb{R}^2$ and which are closed in $\mathbb{R}^2$, giving reasons for your answers:—

   (i) $\{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$,

   (ii) $\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } x + y \leq 1\}$,

   (iii) $\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } x^2 - y^2 = 1\}$.

3. (a) What is a *metric space*?

   (b) What is an *open set* in a metric space? What is a *closed set* in a metric space?

   (c) Prove that any union of open sets in a metric space is an open set. Prove also that any finite intersection of open sets in a metric space is an open set.

   (d) Let d be the distance function on the set $\mathbb{R}$ of all real numbers defined such that $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Show that $\mathbb{R}$, with this distance function, is a metric space. Does there exist a subset of this metric space which is not open? [Justify your answer.]

4. (a) Let X and Y be metric spaces. What is meant by saying that a function $f\colon X \to Y$ is *continuous*?

   (b) What is meant by saying that a sequence $x_1, x_2, x_3, \ldots$ of points in a metric space X *converges* to some point p of X?

   (c) Let F be a closed set in a metric space X. Suppose that some sequence $x_1, x_2, x_3, \ldots$ of points of F converges to some point p of X. Prove that $p \in F$.

   (d) Let X and Y be metric spaces, let $f\colon X \to Y$ be a continuous function from X to Y, and let $x_1, x_2, x_3, \ldots$ be a sequence of points of X that converges to some point p of X. Prove that the sequence $f(x_1), f(x_2), f(x_3), \ldots$ converges to $f(p)$.

   (e) Let $f\colon X \to \mathbb{R}$ be a real-valued function on a metric space X, and let $x_1, x_2, x_3, \ldots$ be a sequence of points of X that converges to some point p of X. Suppose that $|f(x_j)| \leq 1$ for all positive integers j. Explain why $|f(p)| \leq 1$.

5. (a) What is a *topological space*?

   (b) Let $X_1, X_2, \ldots, X_n$ be topological spaces. Give the definition of the *product topology* on the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of $X_1, X_2, \ldots, X_n$, and prove that the collection of open sets in $X_1 \times X_2 \times \cdots \times X_n$ satisfies the axioms in the definition of a topological space.

   (c) Prove that a function $f\colon Z \to X_1 \times X_2 \times \cdots \times X_n$ from a topological space Z to the Cartesian product of topological spaces $X_1, X_2, \ldots, X_n$ is continuous if and only if the components $f_1, f_2, \ldots, f_n$ of f are continuous, where

$$f(z) = (f_1(z), f_2(z), \ldots, f_n(z))$$

   for all $z \in Z$.

6. (a) What is a *compact* topological space?

   (b) Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Prove that f is bounded above and below on X. Prove also that there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.

   (c) Prove that every sequence of points in a compact metric space has a convergent subsequence.

7. (a) What is a *connected* topological space?

   (b) Prove that a topological space X is connected if and only if every continuous function $f\colon X \to \mathbb{Z}$ from X to the set of integers is constant.

   (c) Let X be a topological space. Suppose that $X = A \cup B$, where A and B are connected subsets of X, and $A \cap B$ is non-empty. Prove that X is connected.

   (d) What are the connected components of $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \neq 1\}$? [Justify your answer.]

8. (a) What is a *norm* on a real or complex vector space? What is a *Banach* space?

 (b) Let X be a topological space, and let $C(X, \mathbb{R}^n)$ denote the vector space consisting of all bounded continuous functions from X to $\mathbb{R}^n$. The *supremum norm* $\|f\|$ of such a function f is defined to be the least upper bound (or supremum) of the values of $|f(x)|$ for all points x of X. Prove that $C(X, \mathbb{R}^n)$, with the supremum norm, is a Banach space.

 [You may use without proof the result that a function $f: X \to \mathbb{R}^n$ is continuous if and only if, given any point x of X and given any $\varepsilon > 0$, there exists some open set U_x in X such that $x \in U_x$ and $|f(u) - f(x)| < \varepsilon$ for all $u \in U_x$.]

9. (a) Let $\gamma: [0,1] \to \mathbb{C}$ be a closed curve in the complex plane (where $\gamma(0) = \gamma(1)$), and let w be a complex number that does not lie on the curve. Give the definition of the *winding number* $n(\gamma, w)$ of the closed curve γ about w.

 (b) Let w be a complex number and, for each $\tau \in [0,1]$, let $\gamma_\tau: [0,1] \to \mathbb{C}$ be a closed curve in $\mathbb{C}$ which does not pass through w. Suppose that the map sending $(t, \tau) \in [0,1] \times [0,1]$ to $\gamma_\tau(t)$ is a continuous map from $[0,1] \times [0,1]$ to $\mathbb{C}$. Using the Monodromy Theorem, or otherwise, prove that $n(\gamma_0, w) = n(\gamma_1, w)$.

 (c) Let $f: D \to \mathbb{C}$ be a continuous map defined on the closed unit disk D in $\mathbb{C}$, and let $w \in \mathbb{C} \setminus f(D)$. Prove that $n(f \circ \sigma, w) = 0$, where $\sigma: [0,1] \to \mathbb{C}$ is the parameterization of the unit circle defined by $\sigma(t) = \exp(2\pi i t)$, and $n(f \circ \sigma, w)$ is the winding number of $f \circ \sigma$ about w.

 (d) Prove that there does not exist any continuous map $r: D \to \partial D$ with the property that $r(z) = z$ for all $z \in \partial D$, where ∂D denotes the boundary circle of the closed unit disk D.

 (e) Prove the *Brouwer Fixed Point Theorem in Two Dimensions*, which states that any continuous map $f: D \to D$ from the closed unit disk D to itself has at least one fixed point.