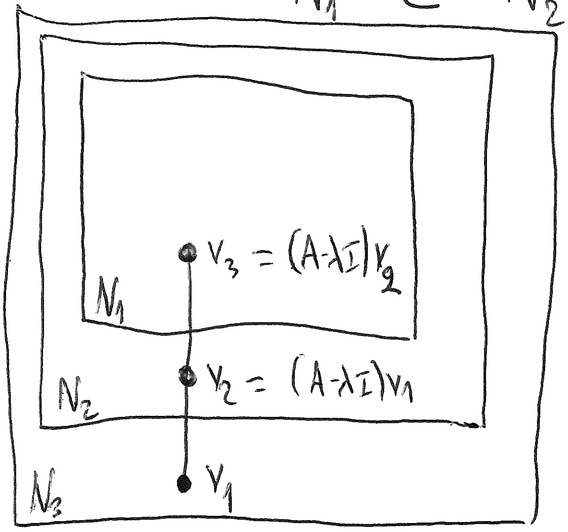
Jordan chains

- Suppose λ is an eigenvalue of the matrix A.

We know the null spaces $N_j = N(A-\lambda I)^j$ are increasing with j and they eventually stabilise. Suppose

$$N_1 \subset N_2 \subset N_3 = N_4, \text{ for instance.}$$



We pick a vector $v_1 \in N_3$ with $v_1 \notin N_2$.

Then $(A-\lambda I)^3 \vec{v}_1 = 0$ but $(A-\lambda I)^2 \vec{v}_1 \neq 0$.

- Define $\vec{v}_2 = (A-\lambda I)v_1$. Then

$$(A-\lambda I)^2 \vec{v}_2 = 0 \quad \text{but} \quad (A-\lambda I)\vec{v}_2 \neq 0$$

which means $\vec{v}_2 \in N_2$ but $\vec{v}_2 \notin N_1$.

- Define $\vec{v}_3 = (A-\lambda I)\vec{v}_2$.

Then $(A-\lambda I)\vec{v}_3 = 0$ so $\vec{v}_3$ is an eigenvector.

__Definition.__ We say that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ form a Jordan chain of length k, if each vector is obtained from the previous one by multiplication with $A-\lambda I$. More precisely, we need $\vec{v}_1, \ldots, \vec{v}_k$ to be nonzero and

$$(A-\lambda I)\vec{v}_i = v_{i+1} \quad \text{if } i<k$$
$$(A-\lambda I)\vec{v}_k = 0.$$

__Equivalently,__ we can pick $\vec{v}_1 \in N(A-\lambda I)^k$ with $\vec{v}_1 \notin N(A-\lambda I)^{k-1}$ and start multiplying to get

$$\vec{v}_2 = (A-\lambda I)\vec{v}_1$$
$$\vec{v}_3 = (A-\lambda I)\vec{v}_2 \quad \text{etc.}$$

This gives vectors $\vec{v}_1, \ldots, \vec{v}_k$ with $(A-\lambda I)\vec{v}_k = 0$, so the last vector $\vec{v}_k$ is an eigenvector with eigenvalue λ.

__Example.__ Suppose A is 5×5 and we have

$v_1, v_2, v_3 \longrightarrow$ a Jordan chain with $\lambda = 3$

$v_4, v_5 \longrightarrow$ a Jordan chain with $\lambda = 2$.

If v_1, v_2, v_3, v_4, v_5 are linearly independent and $B = [v_1\ v_2 \dots v_5]$, then $B^{-1}AB$ can be computed as follows. Recall: $B^{-1}AB\vec{e}_k$ lists coeffs for expressing $A\vec{v}_k$ in terms of $v_1, \dots, v_5$. Namely,

$$B^{-1}AB\vec{e}_k = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix} = \sum x_i \vec{e}_i \iff A\vec{v}_k = \sum x_i \vec{v}_i .$$

__In our case__ v_1, v_2, v_3 is a Jordan chain so

$$\vec{v}_2 = (A-\lambda I)\vec{v}_1 = (A-3I)\vec{v}_1 = A\vec{v}_1 - 3\vec{v}_1$$

$$v_3 = (A-\lambda I)\vec{v}_2 = (A-3I)\vec{v}_2 = A\vec{v}_2 - 3\vec{v}_2$$

and $(A-\lambda I)\vec{v}_3 = 0$ so $A\vec{v}_3 = \lambda\vec{v}_3 = 3\vec{v}_3$.

Similarly, v_4, v_5 is a Jordan chain so

$$\vec{v}_5 = (A-\lambda I)\vec{v}_4 = (A-2I)\vec{v}_4 = Av_4 - 2\vec{v}_4$$

$(A-\lambda I)\vec{v}_5 = 0$ and $A\vec{v}_5 = 2\vec{v}_5$.

__We get__

$$B^{-1}AB = \left[\begin{array}{ccc|cc} 3 & 0 & 0 & \cdot & \cdot \\ 1 & 3 & 0 & \cdot & \cdot \\ 0 & 1 & 3 & \cdot & \cdot \\ \hline 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$Av_2 = 3v_2 + v_3$

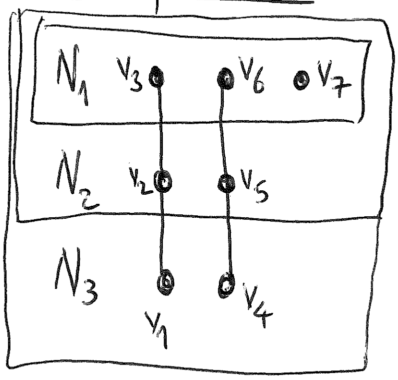
$Av_1 = 3v_1 + v_2$

Definition. A $k \times k$ Jordan block with eigenvalue λ is a $k \times k$ matrix of the form $J = \begin{bmatrix} \lambda & & & \\ 1 & \lambda & & \\ & 1 & \ddots & \\ & & 1 & \lambda \end{bmatrix}$. We are going to show that $B^{-1}AB$ can always be made block diagonal with such blocks. We'll determine the number/sizes of blocks.

Jordan chain diagrams

We can depict the null spaces $N(A-\lambda I)^j = N_j$ using dots for linearly independent vectors.

Example 1. Suppose $\dim N_1 = 3$, $\dim N_2 = 5$, $\dim N_3 = 7$.



We get 3 dots for N_1,

$5-3=2$ additional dots for N_2

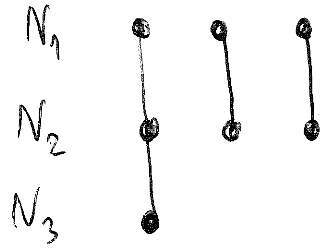
and $7-5=2$ additional dots for N_3.

A vector $v_1 \in N_3$ with $v_1 \notin N_2$ gives rise to a JC of length 3, where $\vec{v}_2 = (A-\lambda I)\vec{v}_1$
$\vec{v}_3 = (A-\lambda I)v_2$.

We actually get 2 Jordan chains of length 3 and also a vector $\vec{v}_7$ which is an eigenvector, a JC of length 1.

These contribute blocks $\begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix}$ and $[\lambda]$.

Example 2. Suppose $\dim N_1 = 3$, $\dim N_2 = 6$, $\dim N_3 = 7$.



In this case, we get

1 JC of length 3

2 JC of length 2.

Those contribute $\begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix}$.

Example 3. Let $A = \begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix}$.

Then $f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$.

Thus $\lambda = 2$ and $N(A-\lambda I) = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

Look at $N(A-\lambda I)^2$. We have $(A-\lambda I)^2 = 0$ and $\dim N(A-\lambda I)^2 = 2$. This gives  and a Jordan chain of length 2.

Next week: we'll show that there is always a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$) consisting of generalised eigenvectors $\vec{v} \in N(A-\lambda I)^j$ for various λ, j.

Jordan chain: consists of nonzero vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ such that

$$(A-\lambda I)\vec{v}_i = \vec{v}_{i+1} \quad \text{for } i<k \qquad \text{and} \qquad (A-\lambda I)\vec{v}_k = 0.$$

Note that $\vec{v}_1 \in N(A-\lambda I)^k$, $\vec{v}_1 \notin N(A-\lambda I)^{k-1}$ and $\vec{v}_k \in N(A-\lambda I)$.

Theorem (Jordan form) Suppose $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ form a basis of $\mathbb{C}^n$ consisting of Jordan chains with several eigenvalues, possibly. Then $B = [\vec{w}_1 \dots \vec{w}_n]$ is invertible and $B^{-1}AB$ is block diagonal with blocks $\begin{bmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}$. We get a $k \times k$ block for each chain of length k and λ corresponds to the eigenvalue of the chain.

Proof. Suppose $\vec{w}_1, \dots, \vec{w}_k$ is the first chain with eigenvalue λ.

Then

$$\begin{aligned}
(A-\lambda I)\vec{w}_1 &= \vec{w}_2 & &\text{so} & A\vec{w}_1 &= \lambda\vec{w}_1 + \vec{w}_2 \\
&\ \vdots & & & &\ \vdots \\
(A-\lambda I)\vec{w}_{k-1} &= \vec{w}_k & &\text{so} & A\vec{w}_{k-1} &= \lambda\vec{w}_{k-1} + \vec{w}_k \\
(A-\lambda I)\vec{w}_k &= 0 & &\text{so} & A\vec{w}_k &= \lambda\vec{w}_k.
\end{aligned}$$

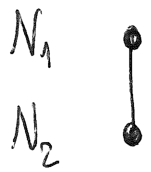
This implies that the first columns of $B^{-1}AB$ are $\begin{bmatrix} \lambda & & \\ 1 & \ddots & \\ & 1 & \lambda \\ \hline 0 & \dots & 0 \end{bmatrix}$.

We thus get a $k \times k$ block for this chain. We proceed similarly with the other chains. $\square$

Example 1. Let $A = \begin{bmatrix} 7 & -8 \\ 2 & -1 \end{bmatrix}$. Then $\lambda = 3, 3$.

We get $A - \lambda I = \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ ... 1 pivot & 1 free
$\dim N(A-\lambda I) = 1$.

Also $(A-\lambda I)^2 = (A-3I)^2 = 0$ is the zero matrix and $\dim N(A-\lambda I)^2 = 2$.



This gives a Jordan chain of length 2 and the Jordan form is $J = \begin{bmatrix} 3 & \\ 1 & 3 \end{bmatrix}$.

To actually find a Jordan chain, we pick a vector $\vec{v}_1 \in N_2$ with $\vec{v}_1 \notin N_1$. Here $N_1 = N(A-\lambda I) = \left\{ \begin{bmatrix} 2y \\ y \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.
Pick any vector $\vec{v}_1$ that is not a scalar multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
We can take $\vec{v}_1 = \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .... $\vec{v}_2 = (A-\lambda I)\vec{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

or $\vec{v}_1 = \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ .... $\vec{v}_2 = (A-\lambda I)\vec{v}_1 = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$.

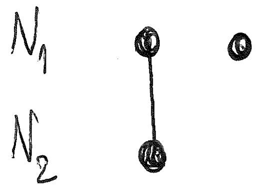
By the general theory $B = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \Rightarrow B^{-1}AB = \begin{bmatrix} 3 & \\ 1 & 3 \end{bmatrix}$.

Example 2. Let $A = \begin{bmatrix} -2 & 3 & 3 \\ -2 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$. Then $\lambda = 1, 1, 1$.

We get $A - \lambda I = \begin{bmatrix} -3 & 3 & 3 \\ -2 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so $\dim N(A-\lambda I) = 2$

Moreover $(A-\lambda I)^2 = 0 =$ zero matrix so $\dim N(A-\lambda I)^2 = 3$.

These dimensions suffice to find the Jordan form.



We get a Jordan chain of length 2 and a Jordan chain of length 1 (eigenvector).

The Jordan form is $J = \left[\begin{array}{cc|c} 1 & & \\ 1 & 1 & \\ \hline & & 1 \end{array}\right]$.

To find the Jordan chain, we need $v_1 \in N_2$ with $v_1 \notin N_1$.

$\boxed{\begin{array}{l}\underline{\text{Theorem}} \text{ (Linear indep. of Jordan chains)} \\ \text{Suppose } \gamma_1, \gamma_2, \dots \gamma_m \text{ are Jordan chains corresponding to the} \\ \text{same eigenvalue. If the } \underline{\text{last}} \text{ vectors in the chains are lin. indep.} \\ \text{then } \underline{\text{all}} \text{ vectors within the chains are lin. indep.}\end{array}}$

Proof. We use induction on k = size of longest chain.
When $k=1$, chains contain one vector each, so the result is clear.
Assume it holds for k and consider chains $\gamma_1, \dots, \gamma_m$ one of which has length $k+1$. Write

$$\gamma_1 = \{\vec{v}_{11}, \vec{v}_{12}, \dots, \vec{v}_{1k_1}\}$$
$$\gamma_2 = \{\vec{v}_{21}, \vec{v}_{22}, \dots, \vec{v}_{2k_2}$$

and

$$\gamma_i = \{\vec{v}_{i1}, \vec{v}_{i2}, \dots, v_{ik_i}\}.$$

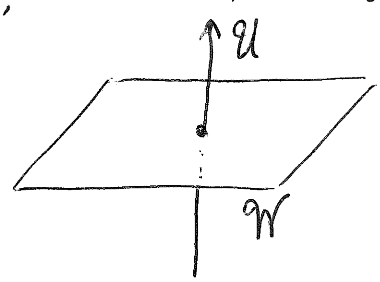
Then $\sum a_{ij} \vec{v}_{ij} = 0$ for some scalars a_{ij}

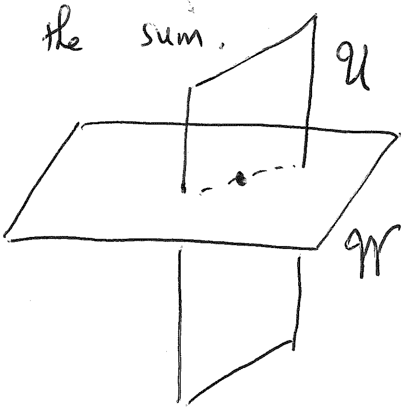
$$\Rightarrow \sum a_{ij} (A - \lambda I) \vec{v}_{ij} = 0$$

$$\Rightarrow \sum a_{ij} \vec{v}_{i,j+1} = 0 \quad \text{with sum not containing last vectors.}$$

By induction, these $a_{ij} = 0$ (length $\leq k$).
so all $a_{ij} = 0$. $\square$

Direct sums Suppose $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then $\mathcal{U}+\mathcal{W} = \{\vec{u}+\vec{w} : \vec{u}\in\mathcal{U} \text{ and } \vec{w}\in\mathcal{W}\}$ is a subspace as well and the sum $\mathcal{U}+\mathcal{W}$ is *direct* if $\mathcal{U}\cap\mathcal{W} = \{0\}$. In that case, we write $\mathcal{U}\oplus\mathcal{W}$ for the sum.





Check $\mathcal{U}+\mathcal{W}$ is a subspace. Namely,

suppose $\vec{z}_1 \in \mathcal{U}+\mathcal{W}$ and $\vec{z}_2 \in \mathcal{U}+\mathcal{W}$.

Then $\vec{z}_1 = \vec{u}_1 + \vec{w}_1$ and $\vec{z}_2 = \vec{u}_2+\vec{w}_2$ with $\vec{u}_1, \vec{u}_2 \in \mathcal{U}$ and $\vec{w}_1,\vec{w}_2\in\mathcal{W}$.

Thus $\vec{z}_1+\vec{z}_2 = (\vec{u}_1+\vec{u}_2)+(\vec{w}_1+\vec{w}_2) \in \mathcal{U}+\mathcal{W}$ as well.

Similarly, $\vec{z}\in\mathcal{U}+\mathcal{W} \Rightarrow \vec{z} = \vec{u}+\vec{w} \Rightarrow \lambda\vec{z} = \lambda\vec{u}+\lambda\vec{w} \in \mathcal{U}+\mathcal{W}$.

Theorem (Direct sums) If the sum $\mathcal{U}\oplus\mathcal{W}$ is direct, then one can get a basis by merging a basis $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$ for $\mathcal{U}$ with a basis $\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ for $\mathcal{W}$.

In particular $\dim(\mathcal{U}\oplus\mathcal{W}) = \dim\mathcal{U} + \dim\mathcal{W}$.

Proof. To show $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m, \vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ span the direct sum, note that $\vec{z}\in\mathcal{U}\oplus\mathcal{W} \Rightarrow \vec{z} = \vec{u}+\vec{w} = \sum c_i\vec{u}_i + \sum x_j\vec{w}_j$. To show they are lin. independent, note that

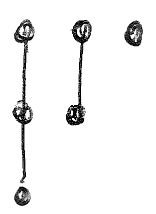
$$\sum c_i\vec{u}_i + \sum x_j\vec{w}_j = 0 \Rightarrow \sum c_i\vec{u}_i = -\sum x_j\vec{w}_j = 0$$

$$\Rightarrow c_i = 0 = x_j \quad \forall i,j. \quad \blacksquare$$

Theorem (Primary decomposition) Let A be $n\times n$. Let $\lambda_1, \dots, \lambda_p$ be its distinct eigenvalues. Let $N(A-\lambda_i I)^{k_i}$ be the point at which the null spaces stabilise. Then $\mathbb{C}^n = N(A-\lambda_1 I)^{k_1} \oplus N(A-\lambda_2 I)^{k_2} \oplus \dots \oplus N(A-\lambda_p I)^{k_p}$.

<u>Primary decomposition:</u> $\mathbb{C}^n = N(A-\lambda_1 I)^{k_1} \oplus N(A-\lambda_2 I)^{k_2} \oplus \cdots \oplus N(A-\lambda_p I)^{k_p}$

whenever A is an $n\times n$ matrix with distinct eigenvalues $\lambda_1, \dots, \lambda_p$ and $k_1, \dots, k_p$ the exponents at which the null spaces $N(A-\lambda_i I)^{k_i}$ are stabilising. This shows we can find a basis $\vec{v}_1, \dots, \vec{v}_n$ consisting of gen. eigenvectors of A. The next step is to find a basis consisting of Jordan chains



$N(A-\lambda I)$ ... $\dim N(A-\lambda I) = 3$

$N(A-\lambda I)^2$ ... $\dim N(A-\lambda I)^2 = 5$

$N(A-\lambda I)^3$ ... $\dim N(A-\lambda I)^3 = 6$

We need a basis that consists of Jordan chains.

<u>Theorem</u> (Number of dots and number of Jordan chains).
Suppose λ is an eigenvalue with multiplicity m, namely suppose that λ is a root of the char. polynomial with multiplicity m.

Then $\boxed{\dim N(A-\lambda I) = \text{number of Jordan chains}}$

and $\boxed{\dim N(A-\lambda I)^k = \text{dimension of largest null space} = \#\text{ of dots}}$

<u>Proof</u>. The key part is to show that $\underline{\dim N(A-\lambda I)^k \leq m}$.
Namely, the dimension of the null spaces cannot exceed the multiplicity. Assume this for the moment. Let $\lambda_1, \lambda_2, \dots, \lambda_p$ be the eigenvalues. Let $m_1, m_2, \dots, m_p$ be their multiplicities. Then primary decomposition gives $\mathbb{C}^n = N(A-\lambda_1 I)^{k_1} \oplus \cdots \oplus N(A-\lambda_p I)^{k_p} \Rightarrow n = \sum_{i=1}^{p} \dim N(A-\lambda_i I)^{k_i}$

$$\leq \sum_{i=1}^{p} m_i = n,$$

the degree of the char. polynomial. Thus equality <u>must</u> hold for all i and we get $\dim N(A-\lambda_i I)^k = m_i$ for all i.

To prove $\dim N(A-\lambda I)^k \leq m$, consider $T: N(A-\lambda I)^k \to N(A-\lambda I)^k$, where $T(\vec{x}) = A\vec{x}$. Suppose $\dim N(A-\lambda I)^k = j$. The matrix of T is $j\times j$. Its char. polynomial has degree j and the only eigenvalue is λ (by primary decomposition). This implies that $j \leq m$. ▨

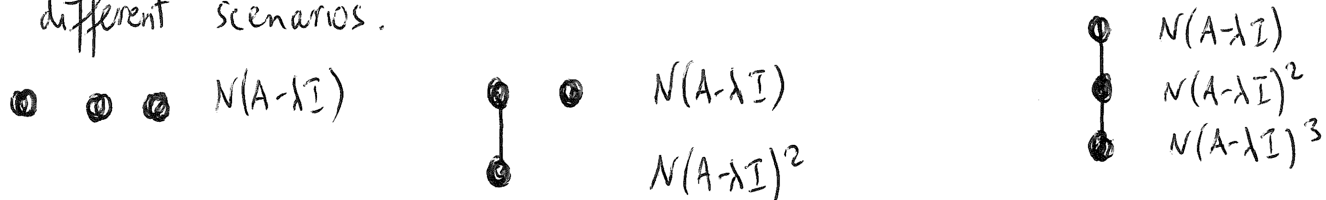
$\boxed{\text{Simple eigenvalue}}$ Suppose λ has multiplicity $m=1$. Then $\dim N(A-\lambda I)^k = 1$ as well. We only get 1 vector in this case!

$\bullet$ Jordan chain diagram has 1 dot (so we don't have to look at higher null spaces).

$\boxed{\text{Double eigenvalue}}$ Suppose λ has multiplicity $m=2$. Then there are two possibilities. <u>Either</u> $\dim N(A-\lambda I) = 2$ .... $\bullet\ \bullet$ --- $\left[\begin{array}{c|c} \lambda & \\ \hline & \lambda \end{array}\right]$

<u>OR</u> $\dim N(A-\lambda I) = 1$ & $\dim N(A-\lambda I)^2 = 2$ --- $\begin{matrix}\bullet\\ |\\ \bullet\end{matrix}$ --- $\begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix}$.

$\boxed{\text{Triple eigenvalue}}$ Suppose λ has multiplicity $m=3$. One can then get 3 different scenarios.



$$\begin{bmatrix} \boxed{\lambda} & & \\ & \boxed{\lambda} & \\ & & \boxed{\lambda} \end{bmatrix} \qquad \left[\begin{array}{cc|c} \lambda & & \\ 1 & \lambda & \\ \hline & & \lambda \end{array}\right] \qquad \begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix}.$$

<u>Remark.</u> Suppose that we have a basis $\vec{v}_1, \dots, \vec{v}_n$ consisting of Jordan chains and suppose we have m of those. Putting those together gives $B = [\vec{v}_1 \dots \vec{v}_n] \Rightarrow B^{-1}AB = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots\ J_m \end{bmatrix}$ with a Jordan block for each chain. We claim that $\dim N(A-\lambda I)$ should be the number of Jordan chains with eigenvalue λ. Now,

$$B^{-1}AB - \lambda I = B^{-1}(A-\lambda I)B = \begin{bmatrix} J_1-\lambda I & & \\ & \ddots & \\ & & J_m-\lambda I \end{bmatrix}.$$

~~If the f~~ Consider the null space of this matrix, say

$$\begin{bmatrix} J_1-\lambda I & & & \\ & J_2-\lambda I & & \\ & & \ddots & \\ & & & J_m-\lambda I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{Then} \quad \begin{aligned} (J_1-\lambda I)\vec{v}_1 &= 0 \\ (J_2-\lambda I)\vec{v}_2 &= 0 \\ &\vdots \\ (J_m-\lambda I)\vec{v}_m &= 0 \end{aligned}$$

We get $(J_i - \lambda I)\vec{v}_i = 0 \Rightarrow \vec{v}_i = 0$ whenever $\lambda \neq$ eigenvalue of J_i

and if $\lambda =$ eigenvalue of J_i, we get $\vec{v}_i \in N(J_i - \lambda I)$.

This implies that $\dim N(B^{-1}AB - \lambda I) = \#$ of blocks with eigenv. λ.

It remains to show $\dim N(B^{-1}AB - \lambda I) = \dim N(A - \lambda I)$.

<u>Example 1.</u> Let $A = \begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix}$. Then $\lambda = 2, 2$.

We look at $A - \lambda I = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$ ... 1 pivot / 1 free
$\dim N(A - \lambda I) = 1$.

This implies $\dim N(A-\lambda I)^2 = 2$ so the JCD is $\begin{matrix}\bullet \\ | \\ \bullet\end{matrix}$ and the

Jordan form is $\begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix} = \begin{bmatrix} 2 & \\ 1 & 2 \end{bmatrix}$.

<u>Example 2.</u> Let $A = \begin{bmatrix} 1 & 0 & 1 \\ & 2 & 0 \\ & & 2 \end{bmatrix}$. Then $\lambda = 1, 2, 2$.

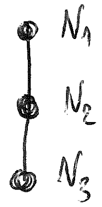
- $\lambda = 1$ contributes $[\lambda] = [1]$.
- $\lambda = 2$ -- $N(A - \lambda I)$ gives $A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$ ... 2 free variables
$\dim N(A - \lambda I) = 2$

We get $\bullet \; \bullet$ and $\left[\begin{array}{c|c} \lambda & \\ \hline & \lambda \end{array}\right] = \left[\begin{array}{c|c} 2 & \\ \hline & 2 \end{array}\right]$.

Example 3. Let $A = \begin{bmatrix} -2 & -7 & 6 \\ 1 & 4 & -2 \\ 0 & 1 & 1 \end{bmatrix}$. Then $\lambda = 1, 1, 1$.

We compute $A - \lambda I \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and so $\dim N(A - \lambda I) = 1$.

This means we have $\dim N(A-\lambda I) = 1$ Jordan blocks.



We can already say $\dim N(A - \lambda I)^2 = 2$, $\dim N(A - \lambda I)^3 = 3$ without working those out. The Jordan form is $J = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}$.

Example 4. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ & 3 & 1 \\ & & 3 \end{bmatrix}$. Then $\lambda = 2, 3, 3$.

For $\lambda = 2$, the simple eigenvalue, we get $[\lambda] = [2]$.

For $\lambda = 3$, the double eigenvalue, we get $A - \lambda I \Rightarrow \begin{bmatrix} -1 & 1 & 1 \\ & 0 & 1 \\ & & 0 \end{bmatrix}$

so $\dim N(A - 3I) = 1$ and we get one chain of length 2. The Jordan form in this case is $J = \begin{bmatrix} \boxed{2} & & \\ & 3 & \\ & 1 & 3 \end{bmatrix}$.

A **Jordan basis** is a basis $v_1 | v_2, v_3$ consisting of Jordan chains.

We have .... $v_1 \in N(A - 2I)$, an eigenvector with $\lambda = 2$

$v_2 \in N(A - 3I)^2$ but $v_2 \notin N(A - 3I)$

$v_3 = (A - \lambda I) v_2 = (A - 3I) v_2$.

If we take $B = [v_1 | v_2 \ v_3]$, then $B^{-1}AB = \left[\begin{array}{c|cc} 2 & & \\ \hline & 3 & \\ & 1 & 3 \end{array}\right]$.

If we take $B = [v_2 \ v_3 | v_1]$, then $B^{-1}AB = \left[\begin{array}{cc|c} 3 & & \\ 1 & 3 & \\ \hline & & 2 \end{array}\right]$.

We normally consider these Jordan forms equivalent (up to perm. of the blocks).

Similar matrices

Theorem We say that A, C are similar when $C = B^{-1}AB$ for some B.

If A, C are similar, then they have the same

(1) char. polynomial (2) eigenvalues (3) dim null space (nullity)

(4) dim column space (rank) (5) determinant and (6) trace.

If A, C are similar, then $(A - \lambda I)^k$, $(C - \lambda I)^k$ are also similar $\forall \lambda, k$.

Proof. ① $\det(C-\lambda I) = \det(B^{-1}AB - \lambda B^{-1}IB) = \cancel{\det B^{-1}} \cdot \det(A-\lambda I) \cdot \cancel{\det B}$

② follows from ①.

③ We relate $N(B^{-1}AB)$ with $N(A)$. Well,

$\vec{x} \in N(B^{-1}AB) \iff B^{-1}AB\vec{x} = 0 \iff A\underbrace{B\vec{x}} = 0 \iff B\vec{x} \in N(A)$.

Suppose $v_1, \dots, v_k$ form a basis for $N(B^{-1}AB)$.
Then $Bv_1, \dots, Bv_k$ are vectors in $N(A)$. It is easy to show that these form a basis of $N(A)$. . . . tutorial.

④ follows from ③ since $\dim N(A) + \dim C(A) = n$.

⑤ $\det(B^{-1}AB) = \cancel{\det B^{-1}} \cdot \det A \cdot \cancel{\det B} = \det A$.

⑥ One has $\underline{tr(AB) = tr(BA)}$ for any matrices A, B.

Namely, $tr(AB) = \sum_i (AB)_{ii} = \sum_i \sum_j A_{ij} B_{ji}$

and $tr(BA) = \sum_j (BA)_{jj} = \sum_j \sum_i B_{ji} A_{ij} \implies tr(AB) = tr(BA)$.

This implies $tr(B^{-1}AB) = tr(AB \cdot B^{-1}) = tr A$.

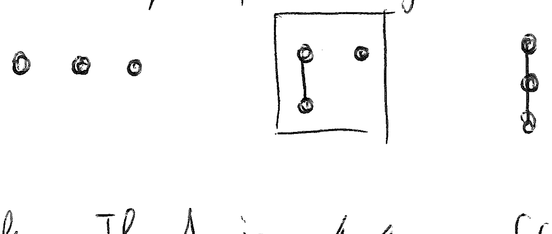
Finally, if A, C are similar and $C = B^{-1}AB$, then

$$\underline{C - \lambda I} = B^{-1}AB - \lambda B^{-1}IB = B^{-1}\,\underline{(A-\lambda I)}\,B.$$

and $(C-\lambda I)^k = B^{-1}(A-\lambda I)^k B$ by induction. ▨

Example. Suppose A is 4×4 with $f(\lambda) = \lambda^3(\lambda-1)$. Suppose $C(A)$ is 2-dim. What is the Jordan form of A?

For $\lambda = 1$, simple eigenvalue, we get $[\lambda] = [1]$.
For $\lambda = 0$, triple eigenvalue, we get $\dim N(A - \lambda I) = \dim N(A) = 2$ chains.



This gives
$$J = \begin{bmatrix} \boxed{1} & & & \\ & 0 & & \\ & 1 & 0 & \\ & & & \boxed{0} \end{bmatrix}$$

Example. If A is 4×4, $f(\lambda) = \lambda^3(\lambda-1)$ and $\dim C(A) = 3$, then the Jordan form is
$$J = \begin{bmatrix} \boxed{1} & & & \\ & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{bmatrix}.$$

Example. If A is 3×3, $f(\lambda) = \lambda^2(\lambda-1)$ and $\dim C(A) = 1$, what is the Jordan form of A^2?

Here, $\lambda = 1$ is simple ... $[1]$

$\lambda = 0$ is double ... $\dim N(A) = 2$ so we get $\left[\begin{array}{c|c} 0 & \\ \hline & 0 \end{array}\right]$.

The Jordan form of A is $J = \left[\begin{array}{c|c|c} 1 & & \\ \hline & 0 & \\ \hline & & 0 \end{array}\right]$ diagonal.

Now, $J = B^{-1}AB$ is similar to A

$\Rightarrow J^2$ is similar to A^2 $\Rightarrow A^2$ is similar to $J^2 = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$

$\Rightarrow A^2$ is diagonalisable and its Jordan form is $\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$.

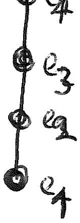
$\boxed{\text{Jordan blocks}}$

Theorem. Suppose $J = \begin{bmatrix} \lambda & & & \\ 1 & \lambda & & \\ & 1 & \ddots & \\ & & 1 & \lambda \end{bmatrix}$ is a $k \times k$ block with eigenvalue λ.

Then ① J has one eigenvalue and one lin. indep. eigenvector

② $(J-\lambda I)^i$ has ones i steps below the diagonal and zeros elsewhere.

③ $N(J-\lambda I)^i = \operatorname{Span}\{\vec{e}_k, \vec{e}_{k-1}, \dots, \vec{e}_{k-i+1}\}$ for each i.



For example, $J-\lambda I = \begin{bmatrix} 0 & & \\ 1 & 0 & \\ & 1 & 0 \end{bmatrix} \Rightarrow (J-\lambda I)^2 = \begin{bmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow (J-\lambda I)^3 = 0$.

Proof. ① J is lower triangular $\Rightarrow \lambda$ is only eigenvalue.
Also, $\dim N(J-\lambda I) = 1$ because $J-\lambda I$ has $k-1$ pivots.

② $J-\lambda I = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & 1 & 0 \end{bmatrix}$ gives $(J-\lambda I)\vec{e}_i = \vec{e}_{i+1}$ if $i<k$ ~~and~~ and $(J-\lambda I)\vec{e}_k = 0$.

Thus $(J-\lambda I)^2 \vec{e}_i = (J-\lambda I)\vec{e}_{i+1} = \vec{e}_{i+2}$
for all $i \le k-2$. This shows that mult. by $J-\lambda I$ shifts the columns once at a time.

③ To find $N(J-\lambda I)^i$, we look at $(J-\lambda I)^i = \begin{bmatrix} 0 & & & \\ 0 & \ddots & & \\ 1 & \ddots & \ddots & \\ & 1 & 0 & 0 \end{bmatrix}$ with the 1s appearing i steps below the diagonal. The corresponding equations are $x_1 = 0$, $x_2 = 0$ and so on. We get a lin. comb. of the last i variables, namely $\operatorname{Span}\{\vec{e}_k, \vec{e}_{k-1}, \dots \vec{e}_{k-i+1}\}$. ▨

Remark. When $J = \begin{bmatrix} \boxed{J_1} & & \\ & \ddots & \\ & & \boxed{J_m} \end{bmatrix}$ contains several blocks, $J-\lambda I = \begin{bmatrix} \boxed{J_1-\lambda I} & & \\ & \ddots & \\ & & \boxed{J_m-\lambda I} \end{bmatrix}$ is block diagonal and $(J-\lambda I)^i = \begin{bmatrix} (J_1-\lambda I)^i & & \\ & \ddots & \\ & & (J_m-\lambda I)^i \end{bmatrix}$ by induction.

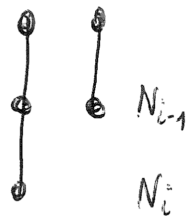
We use this fact & pivot counting to determine number/sizes of Jordan blocks.

$\boxed{\text{Total number of Jordan blocks}}$ Let A be a given matrix, J its Jordan form.

Then $\dim N(A-\lambda I) = \dim N(J-\lambda I)$ by similarity

$= \#$ of Jordan blocks ... because each block contributes 1 free variable

Number of blocks of size $\geq i$



Consider the difference $\dim N(A-\lambda I)^i - \dim N(A-\lambda I)^{i-1}$ $\forall i \geq 2$.

By similarity, that is $\dim N(J-\lambda I)^i - \dim N(J-\lambda I)^{i-1}$

Blocks of size $\leq i-1$ contribute 0 to this difference.

Blocks of size $\geq i$ contribute 1 to this difference.

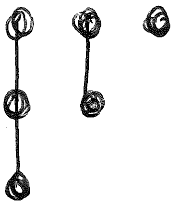
Thus # of blocks of size $\geq i$ is $\dim N(A-\lambda I)^i - \dim N(A-\lambda I)^{i-1}$.

<u>Example</u>. Suppose $\dim N(A-\lambda I) = 3$, $\dim N(A-\lambda I)^2 = 5$, $\dim N(A-\lambda I)^3 = 6$.

We get $\dim N(A-\lambda I) = 3$ Jordan chains

$5-3=2$ Jordan chains of length ≥ 2

$6-5=1$ Jordan chains of length ≥ 3.



This information is provided by the chain diagram.

> **<u>Theorem</u>** (Similarity test) Two matrices A_1, A_2 are similar $\iff$ they have the same Jordan form up to permutation of blocks.

<u>Proof</u>. Suppose $B_1^{-1} \underline{A_1} B_1 = J = B_2^{-1} \underline{\underline{A_2}} B_2$. Then

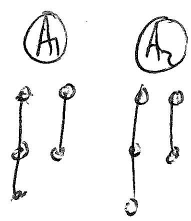
$$A_2 = B_2(B_1^{-1}A_1B_1)B_2^{-1} = (B_2B_1^{-1})A_1(B_1B_2^{-1}) = (B_1B_2^{-1})^{-1}A_1(B_1B_2^{-1}),$$

so A_1, A_2 are similar. Conversely, suppose A_1, A_2 are similar.

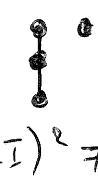
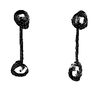
Then $(A_1-\lambda I)^i$ and $(A_2-\lambda I)^i$ are also similar $\forall \lambda, i$.

Thus $\dim N(A_1-\lambda I)^i = \dim N(A_2-\lambda I)^i$ $\forall \lambda, i$.

These determine the number/size of Jordan chains. ☑



<u>Example</u>. Let $A_1 = \left[\begin{array}{ccc|c} 2 & & & \\ 1 & 2 & & \\ & 1 & 2 & \\ \hline & & & 2 \end{array}\right]$ and $A_2 = \left[\begin{array}{cc|cc} 2 & & & \\ 1 & 2 & & \\ \hline & & 2 & \\ & & 1 & 2 \end{array}\right]$.

The diagrams are  and  so A_1, A_2 are <u>not</u> similar.

In fact, $(A_1-2I)^2 \neq 0$ but $(A_2-2I)^2 = 0$.

Powers of a matrix

- Let A be a square matrix and $J = B^{-1}AB$ its Jordan form. To compute A^n, we write $\underline{\underline{J^n}} = (B^{-1}AB)^n = B^{-1}\underline{\underline{A^n}}B$ and conclude that $A^n = B \cdot J^n \cdot B^{-1}$. It remains to find J^n.

- When $J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix}$ consists of several blocks, $J^n = \begin{bmatrix} J_1^n & & \\ & \ddots & \\ & & J_m^n \end{bmatrix}$ by induction. One can prove this using $\left[\begin{array}{c|c} J_1 & \\ \hline & J_2 \end{array}\right]\left[\begin{array}{c|c} J_1 & \\ \hline & J_2 \end{array}\right] = \left[\begin{array}{c|c} J_1^2 & \\ \hline & J_2^2 \end{array}\right]$.
It remains to compute J_i^n for each J_i.

- Consider a $k \times k$ block $J = \begin{bmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}$. When $\lambda = 0$, this is the standard matrix $J = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$ with $J^2 = \begin{bmatrix} 0 & & & & \\ 0 & 0 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 0 \end{bmatrix}$ etc. When $\lambda \neq 0$, we use the binomial theorem

$$(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \quad \text{---- whenever } AB = BA.$$

In our case, we have

$$J^n = (\underbrace{J - \lambda I}_{\substack{\text{standard matrix} \\ \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{bmatrix}}} + \underbrace{\lambda I}_{\substack{\text{diagonal matrix} \\ \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}}})^n = \sum_{k=0}^{n} \binom{n}{k} \underbrace{(J - \lambda I)^k}\, \lambda^{n-k} I.$$

This gives the formula $J^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} (J - \lambda I)^k$.

__Theorem__. Suppose $J = \begin{bmatrix} \lambda & & & \\ 1 & \lambda & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{bmatrix}$ is a $k \times k$ block. Then J^n has entries λ^n on the diagonal, $\binom{n}{1}\lambda^{n-1}$ right below the diagonal, $\binom{n}{2}\lambda^{n-2}$ two steps below the diagonal etc. (as long as $\lambda \neq 0$).

For instance $\begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & \\ n\lambda^{n-1} & \lambda^n \end{bmatrix}$ and $\begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & & \\ n\lambda^{n-1} & \lambda^n & \\ \circledast & n\lambda^{n-1} & \lambda^n \end{bmatrix}$

$\circledast$: $\frac{n(n-1)}{2}\lambda^{n-2}$

and
$$\begin{bmatrix} 2 & & & & \\ 1 & 2 & & & \\ & & 3 & & \\ & & 1 & 3 & \\ & & & & 4 \end{bmatrix}^n = \begin{bmatrix} 2^n & & & & \\ n2^{n-1} & 2^n & & & \\ & & 3^n & & \\ & & n3^{n-1} & 3^n & \\ & & & & 4^n \end{bmatrix}.$$

<u>Example</u>. Take $A = \begin{bmatrix} 8 & -9 \\ 4 & -4 \end{bmatrix}$. We compute A^n.

Then $\lambda = 2, 2$ and $N(A-\lambda I) = N(A-2I) = \text{Span}\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$.

N_1
N_2

This implies $J = \begin{bmatrix} 2 & \\ 1 & 2 \end{bmatrix}$ <u>and</u> $(A-2I)^2 = 0$.

To find a Jordan basis, pick $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in N_2$ with $\vec{v}_1 \notin N_1$,

and let $\vec{v}_2 = (A-\lambda I)\vec{v}_1 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

We get $B = \begin{bmatrix} 1 & 6 \\ 0 & 4 \end{bmatrix} \Rightarrow B^{-1}AB = J = \begin{bmatrix} 2 & \\ 1 & 2 \end{bmatrix}$

$$\Rightarrow (B^{-1}AB)^n = J^n = \begin{bmatrix} 2^n & \\ n2^{n-1} & 2^n \end{bmatrix}$$

$$\Rightarrow B^{-1}A^nB = \begin{bmatrix} 2^n & \\ n2^{n-1} & 2^n \end{bmatrix}$$

$$\Rightarrow A^n = B \cdot \begin{bmatrix} 2^n & \\ n2^{n-1} & 2^n \end{bmatrix} \cdot B^{-1}$$

$$\Rightarrow A^n = \begin{bmatrix} (1+3n)2^n & -9n \cdot 2^{n-1} \\ n \cdot 2^{n+1} & (1-3n) \cdot 2^n \end{bmatrix}.$$

Polynomials & matrices

- Suppose $A^2 = I$. Then the Jordan form J also satisfies $J^2 = I$, because $J^2 = (B^{-1}AB)^2 = B^{-1}A\cancel{B} \cdot \cancel{B^{-1}}AB = B^{-1} \cdot I \cdot B = I$ as well.

More generally, suppose A satisfies some polynomial relation
$$c_n A^n + c_{n-1}A^{n-1} + \dots + c_2A^2 + c_1A + c_0 I = 0.$$
We write this as $f(A) = 0$, where $f(x) = \sum_{k=0}^{n} c_k x^k$.

• $\boxed{\text{Let's check } f(A) = 0 \iff f(J) = 0,}$ where $J =$ Jordan form.

We have
$$f(J) = \sum_{k=0}^{n} c_k J^k = \sum c_k (B^{-1}AB)^k$$
$$= \sum c_k \underline{\underline{B^{-1}}} A^k \underline{\underline{B}}$$
$$= B^{-1}\left(\sum c_k A^k\right)B = B^{-1} f(A) B.$$

This proves $f(A)=0$ if and only if $f(J)=0$.

Example. Suppose A is $n \times n$ with $A^2 = I$. Then $\lambda = \pm 1$ are the only possible eigenvalues.

Namely, $J = B^{-1}AB$ satisfies $J^2 = I$ as well. Since $J = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{bmatrix}$
is lower triangular with diagonal entries λ_i, J^2
is " " " " entries λ_i^2 so $\lambda_i^2 = 1 \Rightarrow \lambda_i = \pm 1$.

$\boxed{\text{Let's check } f(A) = 0 \Rightarrow f(\lambda) = 0 \ \forall \text{ eigenv. } \lambda.}$

We know $\sum c_k A^k = 0$. Let $\vec{v}$ be an eigenvector with $A\vec{v} = \lambda\vec{v}$.

Then $\sum c_k \underbrace{A^k \vec{v}} = 0 \Rightarrow \sum c_k \underbrace{\lambda^k \vec{v}} = 0$ by induction
$$\Rightarrow f(\lambda)\vec{v} = 0$$
$$\Rightarrow f(\lambda) = 0 \text{ or } \vec{v} = 0$$

Since $\vec{v}$ is an eigenvector, we get $f(\lambda) = 0$.

Example. Suppose $A^2 = A$. The eigenvalues satisfy $\lambda^2 = \lambda$
$$\Rightarrow \lambda = 0 \text{ or } \lambda = 1.$$

Next step Every matrix satisfies its char. polynomial $f(\lambda) = \det(A - \lambda I)$.