Jordan chains

Suppose \( \lambda \) is an eigenvalue of the matrix \( A \). We know the null spaces \( N_j = N(A - \lambda I)^j \) are increasing with \( j \) and they eventually stabilise. Suppose

\[
N_1 \subset N_2 \subset N_3 = N_4, \text{ for instance.}
\]

We pick a vector \( \vec{v}_1 \in N_3 \) with \( \vec{v}_1 \not\in N_2 \).

Then \( (A - \lambda I)^3 \vec{v}_1 = 0 \) but \( (A - \lambda I)^2 \vec{v}_1 \neq 0 \).

Define \( \vec{v}_2 = (A - \lambda I) \vec{v}_1 \). Then

\[
(A - \lambda I)^2 \vec{v}_2 = 0 \text{ but } (A - \lambda I) \vec{v}_2 \neq 0
\]

which means \( \vec{v}_2 \in N_2 \) but \( \vec{v}_2 \not\in N_1 \).

Define \( \vec{v}_3 = (A - \lambda I) \vec{v}_2 \).

Then \( (A - \lambda I) \vec{v}_3 = 0 \) so \( \vec{v}_3 \) is an eigenvector.

**Definition.** We say that \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) form a Jordan chain of length \( k \), if each vector is obtained from the previous one by multiplication with \( A - \lambda I \). More precisely, we need \( \vec{v}_1, \ldots, \vec{v}_k \) to be nonzero and

\[
(A - \lambda I) \vec{v}_i = \vec{v}_{i+1} \quad \text{if } i < k
\]

\[
(A - \lambda I) \vec{v}_k = 0.
\]

Equivalently, we can pick \( \vec{v}_1 \in N(A - \lambda I)^k \) with \( \vec{v}_1 \not\in N(A - \lambda I)^{k-1} \) and start multiplying to get

\[
\vec{v}_2 = (A - \lambda I) \vec{v}_1
\]

\[
\vec{v}_3 = (A - \lambda I) \vec{v}_2 \text{ etc.}
\]

This gives vectors \( \vec{v}_1, \ldots, \vec{v}_k \) with \( (A - \lambda I) \vec{v}_k = 0 \), so the last vector \( \vec{v}_k \) is an eigenvector, with eigenvalue \( \lambda \).
Example. Suppose $A$ is $5 \times 5$ and we have

$\begin{align*}
v_1, v_2, v_3 &\rightarrow \text{a Jordan chain with } \lambda = 3 \\
v_4, v_5 &\rightarrow \text{a Jordan chain with } \lambda = 2.
\end{align*}$

If $v_1, v_2, v_3, v_4, v_5$ are linearly independent and $\mathbf{B} = [v_1, v_2, \ldots, v_5]$, then $\mathbf{B}^{-1} \mathbf{A} \mathbf{B}$ can be computed as follows. Recall: $\mathbf{B}^{-1} \mathbf{A} \mathbf{B} \mathbf{e}_k$ lists the coefficients for expressing $A v_k$ in terms of $v_1, \ldots, v_5$. Namely,

$$
\mathbf{B}^{-1} \mathbf{A} \mathbf{B} \mathbf{e}_k = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \sum_i x_i \mathbf{e}_i \quad \Rightarrow \quad A v_k = \sum_i x_i v_i.
$$

In our case $v_1, v_2, v_3$ is a Jordan chain so

$$
\begin{align*}
\tilde{v}_2 &= (A - \lambda I) \tilde{v}_1 = (A - 3I) \tilde{v}_1 = A \tilde{v}_1 - 3 \tilde{v}_1 \\
\tilde{v}_3 &= (A - \lambda I) \tilde{v}_2 = (A - 3I) \tilde{v}_2 = A \tilde{v}_2 - 3 \tilde{v}_2
\end{align*}
$$

and $(A - \lambda I) \tilde{v}_3 = 0$ so $A \tilde{v}_3 = 3 \tilde{v}_3$.

Similarly, $v_4, v_5$ is a Jordan chain so

$$
\begin{align*}
\tilde{v}_5 &= (A - \lambda I) \tilde{v}_4 = (A - 2I) \tilde{v}_4 = A \tilde{v}_4 - 2 \tilde{v}_4 \\
(A - \lambda I) \tilde{v}_5 &= 0 \quad \text{and} \quad A \tilde{v}_5 = 2 \tilde{v}_5.
\end{align*}
$$

We get

$$
\mathbf{B}^{-1} \mathbf{A} \mathbf{B} =
\begin{bmatrix}
3 & 0 & 0 & \cdots & \\
1 & 3 & 0 & \cdots & \\
0 & 1 & 3 & \cdots & \\
0 & 0 & 0 & \cdots & 2 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}
$$

$$
\begin{align*}
A v_2 &= 3v_2 + v_3 \\
A v_4 &= 3v_1 + v_2.
\end{align*}
$$
Definition. A $k \times k$ Jordan block with eigenvalue $\lambda$ is a $k \times k$ matrix of the form $J = \begin{bmatrix} \lambda & 1 \\ & \ddots & 1 \\ & & \lambda \end{bmatrix}$. We are going to show that $B^{-1} A B$ can always be made block diagonal with such blocks. We'll determine the number/sizes of blocks.

**Jordan chain diagrams**

We can depict the null spaces, $N(A - \lambda I)^n = N$, using dots for linearly independent vectors.

**Example 1.** Suppose $\dim N_1 = 3$, $\dim N_2 = 5$, $\dim N_3 = 7$.

We get 3 dots for $N_1$,

$5 - 3 = 2$ additional dots for $N_2$

and $7 - 5 = 2$ additional dots for $N_3$.

A vector $v_1 \in N_3$ with $v_1 \in N_2$ gives rise to a JC of length 3, where $v_2 = (A - \lambda I)v_1$ and $v_3 = (A - \lambda I)v_2$.

We actually get 2 Jordan chains of length 3 and also a vector $v_2$ which is an eigenvector, a JC of length 1.

These contribute blocks $\begin{bmatrix} \lambda & 1 \\ & \ddots & 1 \\ & & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & 1 \\ & \ddots & 1 \\ & & \lambda \end{bmatrix}$ and $\begin{bmatrix} \lambda \\ \vdots \\ \lambda \end{bmatrix}$.

**Example 2.** Suppose $\dim N_1 = 3$, $\dim N_2 = 6$, $\dim N_3 = 7$.

In this case, we get

1 JC of length 3
2 JC of length 2.

Those contribute $\begin{bmatrix} \lambda & 1 \\ & \ddots & 1 \\ & & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda \\ \vdots \\ \lambda \end{bmatrix}$.
Example 3. Let $A = \begin{bmatrix} -4 & 1 \\ -4 & 0 \end{bmatrix}$.
Then $f(\lambda) = \lambda^2 - (b+1)\lambda + \det A = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$.
Thus $\lambda = 2$ and $N(A - \lambda I) = \text{Span} \left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.
Look at $N(A - \lambda I)^2$. We have $(A - \lambda I)^2 = 0$ and $\dim N(A - \lambda I)^2 = 2$. This gives $2$ and a Jordan chain of length $2$.

Next week: We'll show that there is always a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$) consisting of generalised eigenvectors $\vec{v} \in N(A - \lambda I)^k$ for various $k$, $\lambda$.

Jordan chain: consists of nonzero vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ such that

$$(A - \lambda I)\vec{v}_i = \vec{v}_{i+1} \text{ for } i < k \quad \text{and} \quad (A - \lambda I)\vec{v}_k = 0.$$ Note that $\vec{v}_1 \in N(A - \lambda I)^k$, $\vec{v}_k \notin N(A - \lambda I)^{k-1}$ and $\vec{v}_1 \in N(A - \lambda I)$.

Theorem (Jordan form) Suppose $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ form a basis of $\mathbb{C}^n$ consisting of Jordan chains with several eigenvalues, possibly. Then $B = [\vec{w}_1 \ldots \vec{w}_n]$ is invertible and $B^{-1}AB$ is block diagonal with blocks $[\lambda I]$. We get a $k \times k$ block for each chain of length $k$ and $\lambda$ corresponds to the eigenvalue of the chain.

Proof. Suppose $\vec{w}_1, \ldots, \vec{w}_k$ is the first chain with eigenvalue $\lambda$.
Then $(A - \lambda I)\vec{w}_1 = \vec{w}_2$ so $A\vec{w}_1 = \lambda \vec{w}_1 + \vec{w}_2$.

$(A - \lambda I)\vec{w}_k = 0$ so $A\vec{w}_k = \lambda \vec{w}_k$.

This implies that the first columns of $B^{-1}AB$ are $\begin{bmatrix} \lambda & 1 \\ 0 & \ddots & \ddots \end{bmatrix}$. We thus get a $k \times k$ block for this chain. We proceed similarly with the other chains. $\blacksquare$
Example 1. Let \( A = \begin{bmatrix} 7 & -8 \\ 2 & -4 \end{bmatrix} \). Then \( \lambda = 3, 3 \).

We get \( A - \lambda I = \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \) -- 1 pivot & 1 free

\[ \dim N(A-\lambda I) = 1 \]

Also \( (A-\lambda I)^2 = (A-3I)^2 = 0 \) is the zero matrix and \( \dim N(A-\lambda I)^2 = 2 \).

\( N_1 \)

This gives a Jordan chain of length 2

and the Jordan form is \( J = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \).

\( N_2 \)

To actually find a Jordan chain, we pick a vector

\( \vec{v}_1 \in N_2 \) with \( \vec{v}_1 \notin N_1 \). Here \( N_2 = N(A-\lambda I) = \{ \begin{bmatrix} 2y \\ -y \end{bmatrix} \} \) = \{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \} \).

Pick any vector \( \vec{v}_1 \) that is not a scalar multiple of \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

We can take \( \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) ---- \( \vec{v}_2 = (A-\lambda I)\vec{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \)

or \( \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) ---- \( \vec{v}_2 = (A-\lambda I)\vec{v}_1 = \begin{bmatrix} -8 \\ -4 \end{bmatrix} \).

By the general theory \( \beta = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \Rightarrow \beta^{-1}AB = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \).

Example 2. Let \( A = \begin{bmatrix} -2 & 3 & 3 \\ -2 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix} \). Then \( \lambda = 1, 1, 1 \).

We get \( A - \lambda I = \begin{bmatrix} -3 & 3 & 3 \\ -2 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) so \( \dim N(A-\lambda I) = 2 \)

Moreover \( (A-\lambda I)^2 = 0 \) = zero matrix so \( \dim N(A-\lambda I)^2 = 3 \)

These dimensions suffice to find the Jordan form.

\( N_1 \)

We get a Jordan chain of length 2

and a Jordan chain of length 1 (eigenvector).

\( N_2 \)

The Jordan form is \( J = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \).

To find the Jordan chain, we need \( \vec{v}_1 \in N_2 \) with \( \vec{v}_1 \notin N_1 \).
Theorem (Linear indep. of Jordan chains)

Suppose \( \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m \) are Jordan chains corresponding to the same eigenvalue. If the last vectors in the chains are lin. indep., then all vectors within the chains are lin. indep.

Proof. We use induction on \( k = \text{size of longest chain} \).

When \( k=1 \), chains contain one vector each, so the result is clear.

Assume it holds for \( k \) and consider chains \( \vec{x}_1, \ldots, \vec{x}_m \) one of which has length \( k+1 \). Write \( \vec{x}_1 = \{ \vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_{k+1}} \} \)

\( \vec{x}_2 = \{ \vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_{k+2}} \} \)

and \( \vec{x}_i = \{ \vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_{k+1}} \} \).

Then

\[
\sum a_{ij} \vec{v}_{ij} = 0 \quad \text{for some scalars } a_{ij}.
\]

\[
\Rightarrow \sum a_{ij} (A - \lambda I) \vec{v}_{ij} = 0
\]

\[
\Rightarrow \sum a_{ij} \vec{v}_{i,j+1} = 0 \quad \text{with sum not containing last vectors}
\]

By induction, these \( a_{ij} = 0 \) (length \( \leq k \)).

so all \( a_{ij} = 0 \). \( \square \)
**Direct sums**

Suppose \( U \) and \( W \) are subspaces of \( \mathbb{R}^n \) (or \( \mathbb{C}^n \)). Then \( U + W = \{ \vec{u} + \vec{w} : \vec{u} \in U \text{ and } \vec{w} \in W \} \) is a subspace as well and the sum \( U + W \) is **direct** if \( U \cap W = \{0\} \). In that case, we write \( U \oplus W \) for the sum.

Check \( U + W \) is a subspace. Namely, suppose \( \vec{z}_1 \in U + W \) and \( \vec{z}_2 \in U + W \).

Then \( \vec{z}_1 = \vec{u}_1 + \vec{w}_1 \) and \( \vec{z}_2 = \vec{u}_2 + \vec{w}_2 \), with \( \vec{u}_i, \vec{w}_i \in U \) and \( \vec{w}_i, \vec{w}_i \in W \).

Thus \( \vec{z}_1 + \vec{z}_2 = (\vec{u}_1 + \vec{w}_1) + (\vec{u}_2 + \vec{w}_2) \in U + W \) as well.

Similarly, \( \lambda \vec{z} \in U + W \Rightarrow \vec{z} = \vec{u} + \vec{w} \Rightarrow \lambda \vec{z} = \lambda \vec{u} + \lambda \vec{w} \in U + W \).

**Theorem (Direct sums)** If the sum \( U \oplus W \) is direct, then one can get a basis by merging a basis \( \vec{u}_1, \vec{u}_2, ..., \vec{u}_m \) for \( U \) with a basis \( \vec{w}_1, \vec{w}_2, ..., \vec{w}_n \) for \( W \). In particular \( \dim(U \oplus W) = \dim U + \dim W \).

**Proof.** To show \( \vec{u}_1, \vec{u}_2, ..., \vec{u}_m, \vec{w}_1, \vec{w}_2, ..., \vec{w}_n \) span the direct sum, note that \( \vec{z} \in U \oplus W \Rightarrow \vec{z} = \vec{u} + \vec{w} = \sum \lambda_i \vec{u}_i + \sum \lambda_j \vec{w}_j \). To show they are lin. independent, note that

\[
\sum \lambda_i \vec{u}_i + \sum \lambda_j \vec{w}_j = \vec{0} \Rightarrow \sum \lambda_i \vec{u}_i = -\sum \lambda_j \vec{w}_j = \vec{0} \Rightarrow \lambda_i = 0 = \lambda_j \quad \forall i, j.
\]

**Theorem (Primary decomposition)** Let \( A \) be \( n \times n \). Let \( \lambda_1, ..., \lambda_p \) be its distinct eigenvalues. Let \( N(A - \lambda_i I)^k \) be the point at which the null spaces stabilise. Then \( \mathbb{C}^n = N(A - \lambda_1 I)^k \oplus N(A - \lambda_2 I)^k \oplus ... \oplus N(A - \lambda_p I)^k \).
Primary decomposition: 

\[ C^n = N(A-\lambda_1 I)^{k_1} \oplus N(A-\lambda_2 I)^{k_2} \oplus \cdots \oplus N(A-\lambda_p I)^{k_p} \]

whenever \( A \) is an \( n \times n \) matrix with distinct eigenvalues \( \lambda_1, \ldots, \lambda_p \) and \( k_1, \ldots, k_p \) the exponents at which the null spaces \( N(A-\lambda_i I)^k \) are stabilising. This shows we can find a basis \( \vec{v}_1, \ldots, \vec{v}_n \) consisting of gen. eigenvectors of \( A \). The next step is to find a basis consisting of Jordan chains.

\[
\begin{align*}
N(A-\lambda_1 I) & : \quad \dim N(A-\lambda_1 I) = 3 \\
N(A-\lambda_2 I)^2 & : \quad \dim N(A-\lambda_2 I)^2 = 5 \\
N(A-\lambda_3 I)^3 & : \quad \dim N(A-\lambda_3 I)^3 = 6
\end{align*}
\]

**Theorem (Number of dots and number of Jordan chains).**

Suppose \( \lambda \) is an eigenvalue with multiplicity \( m \), namely suppose that \( \lambda \) is a root of the char. polynomial with multiplicity \( m \).

Then

\[
\dim N(A-\lambda I) = \text{number of Jordan chains}
\]

and

\[
\dim N(A-\lambda I)^k = \text{dimension of largest null space} = \# \text{ of dots}.
\]

**Proof.** The key part is to show that \( \dim N(A-\lambda I)^k \leq m \).

Namely, the dimension of the null spaces cannot exceed the multiplicity. Assume this for the moment. Let \( \lambda_1, \lambda_2, \ldots, \lambda_p \) be the eigenvalues. Let \( m_1, m_2, \ldots, m_p \) be their multiplicities. Then primary decomposition gives

\[ C^n = N(A-\lambda_1 I)^{k_1} \oplus \cdots \oplus N(A-\lambda_p I)^{k_p} \Rightarrow n = \sum_{i=1}^{p} \dim N(A-\lambda_i I)^{k_i} \leq \sum_{i=1}^{p} m_i = n, \]

the degree of the char. polynomial. Thus equality must hold for all \( i \) and we get \( \dim N(A-\lambda_i I)^k = m_i \) for all \( i \).

To prove \( \dim N(A-\lambda I)^k \leq m \), consider \( T : N(A-\lambda I)^k \to N(A-\lambda I)^k \), where \( T(x) = Ax \). Suppose \( \dim N(A-\lambda I)^k = j \). The matrix of \( T \) is \( j \times j \).

Its char. polynomial has degree \( j \) and the only eigenvalue is \( \lambda \) (by primary decomposition). This implies that \( j \leq m \).
Suppose $\lambda$ has multiplicity $m = 1$. Then \( \dim N(A - \lambda I) = 1 \) as well. We only get 1 vector in this case!

- Jordan chain diagram has 1 dot (so we don’t have to look at higher null spaces).

**Double eigenvalue** Suppose $\lambda$ has multiplicity $m = 2$. Then there are two possibilities.

Either \( \dim N(A - \lambda I) = 2 \) .......

\[
\begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}
\]

OR \( \dim N(A - \lambda I) = 1 \) & \( \dim N(A - \lambda I)^2 = 2 \) .......

\[
\begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}
\]

**Triple eigenvalue** Suppose $\lambda$ has multiplicity $m = 3$. One can then get 3 different scenarios.

- \( \dim N(A - \lambda I) \)
- \( \dim N(A - \lambda I)^2 \)
- \( \dim N(A - \lambda I)^3 \)

\[
\begin{bmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
0 & 0 & \lambda
\end{bmatrix}
\]

**Remark.** Suppose that we have a basis \( \vec{v}_1, \ldots, \vec{v}_n \) consisting of Jordan chains and suppose we have $m$ of these. Putting those together gives

\[
B = [\vec{v}_1 \ldots \vec{v}_n] \Rightarrow B^{-1}AB = \begin{bmatrix}
J_1 & & \\
& \ddots & \\
& & J_m
\end{bmatrix}
\]

with a Jordan block for each chain. We claim that \( \dim N(A - \lambda I) \) should be the number of Jordan chains with eigenvalue $\lambda$. Now,

\[
B^n AB - \lambda I = B^n (A - \lambda I) B = \begin{bmatrix}
J_1 - \lambda I & & \\
& \ddots & \\
& & J_m - \lambda I
\end{bmatrix}
\]

Consider the null space of this matrix, say

\[
\begin{bmatrix}
J_1 - \lambda I \\
J_2 - \lambda I \\
\vdots \\
J_m - \lambda I
\end{bmatrix}
\begin{bmatrix}
\vec{v}_1 \\
\vec{v}_2 \\
\vdots \\
\vec{v}_m
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Then

\[
(J_1 - \lambda I) \vec{v}_1 = 0 \\
(J_2 - \lambda I) \vec{v}_2 = 0 \\
(J_m - \lambda I) \vec{v}_m = 0
\]
We get \((J_i - \lambda I)\mathbf{v}_i = 0 \Rightarrow \mathbf{v}_i = 0\) whenever \(\lambda \neq\) eigenvalue of \(J_i\) and if \(\lambda =\) eigenvalue of \(J_i\), we get \(\mathbf{v}_i \in N(J_i - \lambda I)\). This implies that \(\dim N(B^2 - AB - \lambda I) = \#\) of blocks with eigen \(\lambda\).

It remains to show \(\dim N(B^2 - AB - \lambda I) = \dim N(A - \lambda I)\).

**Example 1.** Let \(A = \begin{bmatrix} 4 & 1 \\ -4 & 0 \end{bmatrix}\). Then \(\lambda = 2, 2\).

We look at \(A - \lambda I = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\) --- 1 pivot / 1 free \(\dim N(A - \lambda I) = 1\).

This implies \(\dim N(A - \lambda I)^2 = 2\) so the JCD is \(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\) and the Jordan form is \(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\).

**Example 2.** Let \(A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}\). Then \(\lambda = 1, 2, 2\).

\(\lambda = 1\) contributes \(N(\lambda I) = \begin{bmatrix} 1 \end{bmatrix}\).

\(\lambda = 2\) --- \(N(A - \lambda I)\) gives \(A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\) --- 2 free variables \(\dim N(A - \lambda I) = 2\).

We get \(\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}\) and \(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\).
Example 3. Let \( A = \begin{bmatrix} -2 & -7 & 6 \\ 4 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \). Then \( \lambda = 1, 1, 1 \).

We compute \( A-\lambda I \rightarrow \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \) and so \( \dim N(A-\lambda I) = 1 \).

This means we have \( \dim N(A-\lambda I) = 1 \) Jordan blocks.

We can already say \( \dim N(A-\lambda I)^2 = 2 \), \( \dim N(A-\lambda I)^3 = 3 \) without working those out. The Jordan form is \( J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \).

Example 4. Let \( A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & 0 \\ 3 & 3 & 3 \end{bmatrix} \). Then \( \lambda = 2, 3, 3 \).

For \( \lambda = 2 \), the simple eigenvalue, we get \( [\lambda] = [2] \).

For \( \lambda = 3 \), the double eigenvalue, we get \( A-\lambda I = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \) so \( \dim N(A-\lambda I) = 1 \) and we get a chain of length 2. The Jordan form in this case is \( J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \).

A Jordan basis is a basis \( v_1, v_2, v_3 \) consisting of Jordan chains.

We have \( v_1 \in N(A-2I) \), an eigenvector with \( \lambda = 2 \).

\( v_2 \in N(A-3I)^2 \) but \( v_2 \notin N(A-3I) \).

\( v_3 = (A-\lambda I)v_2 = (A-3I)v_2 \).

If we take \( B = [v_1 \ v_2 \ v_3] \), then \( B^{-1}AB = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \).

If we take \( B = [v_2 \ v_3 \ v_1] \), then \( B^{-1}AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \).

We normally consider these Jordan forms equivalent (up to perm. of the blocks).

**Similar matrices**

**Theorem** We say that \( A, C \) are similar when \( C = B^{-1}AB \) for some \( B \).

\( \blacksquare \) If \( A, C \) are similar, then they have the same

1. char. polynomial
2. eigenvalues
3. \( \dim \) null space (nullity)
4. \( \dim \) column space (rank)
5. determinant
6. trace

If \( A, C \) are similar, then \( (A-\lambda I)^k, (C-\lambda I)^k \) are also similar \( \forall k \).
Proof.  1. \[ \det (C - \lambda I) = \det (B^\text{T} A B - \lambda B^\text{T} B) = \det B^\text{T} \cdot \det (A - \lambda I) \cdot \det B \]
2. follows from 1.
3. We relate \( N(\cdot) \) with \( N(A) \). Well,
\[ x \in \text{null}(\cdot) \quad \Rightarrow \quad B^\text{T} A B x = 0 \quad \Rightarrow \quad A B x = 0 \quad \Rightarrow \quad B x \in N(A). \]
Suppose \( x, y, \ldots, v_k \) form a basis for \( N(B^\text{T} A B) \).
Then \( B x, \ldots, B y_k \) are vectors in \( N(A) \). It is easy to show
that these form a basis of \( N(A) \). tutorial.
4. follows from 3. since \( \dim N(A) + \dim \text{null}(C) = n \).
5. \[ \det (B^\text{T} A B) = \det B^\text{T} \cdot \det A \cdot \det B = \det A. \]
6. One has \( \text{tr}(AB) = \text{tr}(BA) \) for any matrices \( A, B \).

Namely, \[ \text{tr}(AB) = \sum_i \sum_j A_{ij} B_{ji}, \]
and \[ \text{tr}(BA) = \sum_i \sum_j B_{ij} A_{ji}, \]
\[ \Rightarrow \text{tr}(AB) = \text{tr}(BA). \]
This implies \( \text{tr}(B^\text{T} A B) = \text{tr}(A B \cdot B^\text{T}) = \text{tr}A. \)

Finally, if \( A, C \) are similar and \( C = B^\text{T} A B \), then
\[ C - \lambda I = B^\text{T} A B - \lambda B^\text{T} I B = B^\text{T} (A - \lambda I) B, \]
and \( (C - \lambda I)^k = B^\text{T} (A - \lambda I)^k B \) by induction.

Example. Suppose \( A \) is \( 4 \times 4 \) with \( f(\lambda) = \lambda^3 (\lambda - 1) \). Suppose \( \text{null}(A) \)
is 2-dim. What is the Jordan form of \( A \)?
For \( \lambda = 1 \), simple eigenvalue, we get \( [1] \).
For \( \lambda = 0 \), triple eigenvalue, we get \( \dim \text{null}(A - \lambda I) = \dim N(A - \lambda I) = 2 \) chains,
\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
This gives \( J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \).

Example. If \( A \) is \( 4 \times 4 \), \( f(\lambda) = \lambda^3 (\lambda - 1) \) and \( \dim \text{null}(\cdot) = 3 \),
then the Jordan form is \( J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \).
Example. If $A$ is $3 \times 3$, $f(A) = X^2 (X-1)$ and $\dim C(A) = 1$, what is the Jordan form of $A^2$?

Here, $\lambda=1$ is simple ... \([1]\)

$\lambda=0$ is double ... $\dim N(A) = 2$ so we get \([0\ 1\ 0] \) diagonal.

The Jordan form of $A$ is $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ diagonal.

Now, $J = B^{-1}AB$ is similar to $A$

$\Rightarrow J^2$ is similar to $A^2$ $\Rightarrow A^2$ is similar to $J^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow A^2$ is diagonalizable and its Jordan form is \([0\ 0\ 0] \)
**Jordan blocks**

**Theorem.** Suppose \( J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \) is a \( k \times k \) block with eigenvalue \( \lambda \).

Then
1. \( J \) has one eigenvalue and one lin. indep. eigenvector.
2. \((J-\lambda I)^i \) has ones \( i \) steps below the diagonal and zeros elsewhere.
3. \( N(J-\lambda I)^i = \text{Span} \{ \overrightarrow{e_k}, \overrightarrow{e_{k-1}}, \ldots, \overrightarrow{e_{k-i}} \} \) for each \( i \).

For example, \( J-\lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow (J-\lambda I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow (J-\lambda I)^3 = 0 \).

**Proof.**

1. \( J \) is lower triangular \( \Rightarrow \lambda \) is only eigenvalue.
   Also, \( \dim N(J-\lambda I) = 1 \) because \( J-\lambda I \) has \( k-1 \) pivots.
2. \( J-\lambda I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) gives \((J-\lambda I) \overrightarrow{e_i} = \overrightarrow{e_{i+1}} \) if \( i < k \) and \((J-\lambda I) \overrightarrow{e_k} = 0 \).

Thus \((J-\lambda I)^2 \overrightarrow{e_i} = (J-\lambda I) \overrightarrow{e_{i+1}} = \cdots = \overrightarrow{e_{i+2}} \) for all \( i < k - 2 \). This shows that mult.
   by \( J-\lambda I \) shifts the columns once at a time.

3. To find \( N(J-\lambda I)^i \), we look at \((J-\lambda I)^i = \begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \) with
   the 1s appearing \( i \) steps below the diagonal. The corresponding equations are \( x_1 = 0 \), \( x_2 = 0 \) and so on. We get a lin. comb. of the last \( i \) variables, namely \( \text{Span} \{ \overrightarrow{e_k}, \overrightarrow{e_{k-1}}, \ldots, \overrightarrow{e_{k-i}} \} \).

**Remark.** When \( J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \) contains several blocks, \( J-\lambda I = \begin{bmatrix} J_1-\lambda I \\ \vdots \\ J_m-\lambda I \end{bmatrix} \) is block diagonal and \( (J-\lambda I)^i = \begin{bmatrix} (J_1-\lambda I)^i \\ \vdots \\ (J_m-\lambda I)^i \end{bmatrix} \) by induction.

We use this fact & pivot counting to determine number/size of Jordan blocks.

**Total number of Jordan blocks**

Let \( A \) be a given matrix, \( J \) its Jordan form.

Then \( \dim N(A-\lambda I) = \dim N(J-\lambda I) \) by similarity

\[ = \# \text{ of Jordan blocks} \]

because each block contributes a free variable.
Consider the difference \( \dim N(A - \lambda I)^i - \dim N(A - \lambda I)^{i-1} \) \( \forall i \geq 2 \). By similarity, that is \( \dim N(J - \lambda I)^i - \dim N(J - \lambda I)^{i-1} \), \( \forall i \geq 0 \).

Blocks of size \( i-1 \) contribute 0 to this difference. Blocks of size \( i \) contribute 1 to this difference. Thus \( \# \) of blocks of size \( i \) is \( \dim N(A - \lambda I)^i - \dim N(A - \lambda I)^{i-1} \).

**Example.** Suppose \( \dim N(A - \lambda I) = 3 \), \( \dim N(A - \lambda I)^2 = 5 \), \( \dim N(A - \lambda I)^3 = 6 \).

We get \( \dim N(A - \lambda I) = 3 \) Jordan chains

\[
5 - 3 = 2 \quad \text{Jordan chains of length } > 2
\]
\[
6 - 5 = 1 \quad \text{Jordan chains of length } \geq 3 .
\]

This information is provided by the chain diagram.

**Theorem (Similarity Test).** Two matrices \( A_1, A_2 \) are similar \( \iff \) they have the same Jordan form up to permutation of blocks.

**Proof.** Suppose \( B_1^{-1} A_1 B_1 = J = B_2^{-1} A_2 B_2 \). Then

\[
A_2 = B_2 (B_1^{-1} A_1 B_1) B_2^{-1} = (B_2 B_1^{-1}) A_1 (B_1 B_2) = (B_1 B_2^{-1}) A_1 (B_1 B_2^{-1}),
\]

so \( A_1, A_2 \) are similar. Conversely, suppose \( A_1, A_2 \) are similar. Then \( (A_1 - \lambda I)^i \) and \( (A_2 - \lambda I)^i \) are also similar \( \forall i \).

Thus \( \dim N(A_1 - \lambda I)^i = \dim N(A_2 - \lambda I)^i \) \( \forall i \).

These determine the number/sizes of Jordan chains.

**Example.** Let \( A_1 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \) and \( A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \).

The diagrams are \( \bullet \circ \) and \( \bullet \circ \circ \) so \( A_1, A_2 \) are not similar.

In fact, \( (A_1 - 2 \mathbf{I})^2 \neq 0 \) but \( (A_2 - 3 \mathbf{I})^2 = 0 \).
Powers of a matrix

- Let $A$ be a square matrix and $J = B^{-1}AB$ its Jordan form. To compute $A^n$, we write $J^n = (B^{-1}AB)^n = B^{-1}A^nB$ and conclude that $A^n = B^{-1}J^nB$. It remains to find $J^n$.

- When $J = \begin{bmatrix} J_1 & 0 \\ 0 & J_m \end{bmatrix}$ consists of several blocks, $J^n = \begin{bmatrix} J_1^n & 0 \\ 0 & J_m^n \end{bmatrix}$ by induction. One can prove this using $\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} = \begin{bmatrix} J_1^2 & 0 \\ 0 & J_2 \end{bmatrix}$. It remains to compute $J_i^n$ for each $J_i$.

- Consider a $k \times k$ block $J = \begin{bmatrix} \lambda I_k \\ 0 \end{bmatrix}$ when $\lambda = 0$, this is the standard matrix $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $J^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ etc. When $\lambda \neq 0$, we use the binomial theorem
  \[(A + B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} \quad \text{whenever } AB = BA.
  \]

In our case, we have
  \[J^n = (J - \lambda I + \lambda I)^n = \sum_{k=0}^{n} \binom{n}{k} (J - \lambda I)^k \lambda^{n-k} I.
  \]

This gives the formula $J^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k} (J - \lambda I)^k$.

**Theorem.** Suppose $J = \begin{bmatrix} \lambda^1 \\ & \ddots \\ && \lambda \end{bmatrix}$ is a $k \times k$ block. Then $J^n$ has entries $\lambda^n$ on the diagonal, $\binom{n}{1} \lambda^{n-1}$ right below the diagonal, $\binom{n}{2} \lambda^{n-2}$ two steps below the diagonal etc. (as long as $\lambda \neq 0$).

For instance $\begin{bmatrix} \lambda & 2 \\ & \ddots \\ && \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ \vdots & \ddots \end{bmatrix}$ and $\begin{bmatrix} \lambda & 1 \\ & \ddots \end{bmatrix}^n = \begin{bmatrix} \lambda^n & \lambda^{n-1} \\ & \ddots \end{bmatrix}$.
Example. Take $A = \begin{bmatrix} 8 & -9 \\ 4 & -4 \end{bmatrix}$. We compute $A^n$.

Then $\lambda = 2, 2$ and $N(A - \lambda I) = N(A - \lambda I) = \text{Span} \{ [3] \}$.

This implies $J = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and $(A - \lambda I)^2 = 0$.

To find a Jordan basis, pick $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in N_2$ with $v_1 \notin N_1$.

and set $v_2 = (A - \lambda I) v_1 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$.

We get $B = \begin{bmatrix} 1 & 6 \\ 0 & 4 \end{bmatrix}$ $\Rightarrow$ $B^{-1} A B = J = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow (B^{-1} A B)^n = J^n = \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix}$

$\Rightarrow B^{-1} A^n B = \begin{bmatrix} 2^n \\ 2^n \end{bmatrix}$

$\Rightarrow A^n = B \cdot \begin{bmatrix} 2^n \\ 2^n \end{bmatrix} \cdot B^{-1}$

$\Rightarrow A^n = \begin{bmatrix} (1 + 3n) 2^n & -9n \cdot 2^n \\ n \cdot 2^n & (1 - 3n) 2^n \end{bmatrix}$

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### Polynomials & Matrices

- Suppose $A^2 = I$. Then the Jordan form $J$ also satisfies $J^2 = I$, because $J^2 = (B^{-1} A B)^2 = B^{-1} A^2 B$. $B^{-1} A B = B^{-1} I B = I$ as well.

More generally, suppose $A$ satisfies some polynomial relation $c_n A^n + c_{n-1} A^{n-1} + \ldots + c_2 A^2 + c_1 A + c_0 I = 0$.

We write this as $f(A) = 0$, where $f(x) = \sum_{k=0}^{n} c_k x^k$. 
Let's check \( f(A) = 0 \) \( \Rightarrow \) \( f(\lambda) = 0 \) for every eigenvalue \( \lambda \).

We know \( \sum c_k A^k = 0 \). Let \( \vec{v} \) be an eigenvector with \( A \vec{v} = \lambda \vec{v} \).

Then \( \sum c_k A^k \vec{v} = 0 \) \( \Rightarrow \sum c_k \lambda^k \vec{v} = 0 \) by induction.

\( \Rightarrow f(\lambda) \vec{v} = 0 \)

\( \Rightarrow f(\lambda) = 0 \) or \( \vec{v} = 0 \)

Since \( \vec{v} \) is an eigenvector, we get \( f(A) = 0 \).

Example. Suppose \( A^2 = A \). The eigenvalues satisfy \( \lambda^2 = \lambda \)

\( \Rightarrow \lambda = 0 \) or \( \lambda = 1 \).

Next, every matrix satisfies its char. polynomial \( f(A) = \det(A - \lambda I) \).