Spectral theorem. Suppose $A$ is real and symmetric. Then there is an orthogonal matrix $B$ such that $B^t A B = B^{-1} A B = \text{diagonal}$. 

Proof. We use induction. If $A = [a]$ is $1 \times 1$, then $B = [1]$ will do since $B^t A B = [a]$ is diagonal. Assume the result for $n \times n$. Let $A$ be $(n+1) \times (n+1)$ and pick an eigenvector $\hat{v}_1$ with eigenvalue $\lambda_1$, say. Extend $\{v_1\}$ to a basis and use Gram-Schmidt to get an orthonormal basis $w_1, w_2, \ldots, w_n$ with $w_1 = \frac{1}{\|v_1\|} \hat{v}_1$.

Thus $w_1$ is an eigenvector as well.

We claim $B^t A B = B^{-1} A B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1 \end{bmatrix}$ when $B = [w_1 \ldots w_n]$.

In fact, we know $B^t = B^{-1}$ by orthonormality and also $(B^t A B) e_1 = B^t A w_1 = B^t (\lambda_1 w_1) = B^t (\lambda_1 e_1) = \lambda_1 e_1$, while $B^t A B$ is symmetric since $(B^t A B)^t = B^t A^t B^t = B^t A B$.

Then $P$ is symmetric as well, so we can use the induction hypothesis to get $Q$ orthogonal with $Q^t P Q = \text{diagonal}$. This implies that

$$
\begin{bmatrix} 1 & \cdots & \cdots & 1 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\
Q & \cdots & \cdots & Q
\end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1 \end{bmatrix}
$$

so $R^t B^t A B R = \text{diagonal}$

so $(BR)^t A (BR) = \text{diagonal}$. It remains to check $BR$ is orthogonal. In fact,

$$(BR)^t BR = R^t B^t BR = R^t R \text{ by above}$$

and $R$ is orthogonal (since the columns of $R$ are mutually orthogonal unit vectors).
Application (Quadratic forms) Consider a quadratic function \( Q \) in \( n \) variables \( Q(x_1, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j \). This arises from bilinear forms \( \langle \vec{x}, \vec{y} \rangle = \sum_{i \leq j} a_{ij} x_i y_j \) by taking \( \vec{x} = \vec{y} \).

We need to be careful with the off-diagonal terms:

\[
\begin{align*}
\quad a_{11} x_1 y_1 + a_{12} x_1 y_2 + a_{21} x_2 y_1 + a_{22} x_2 y_2 \\
\rightarrow a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22} x_2^2
\end{align*}
\]

Thus, \( Q(x_1, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j = \vec{x}^t A \vec{x} \) with

\[
\begin{align*}
\quad a_{ij} &= \begin{cases} 
\frac{1}{2} \text{ well of } x_i x_j & \text{if } i \neq j \\
\frac{1}{2} \text{ well of } x_i^2 & \text{if } i = j
\end{cases}
\end{align*}
\]

Example. Consider the quadratic equation \( xy = 1 \).

We can write \( Q(x, y) = xy = \vec{x}^t A \vec{x} \) with \( \vec{x} = [x \ y] \) and \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

Since \( A \) is symmetric, \( A \) is diagonalizable by the spectral theorem.

We have eigenvalues \( \gamma_1 = 1/2 \) and \( \gamma_2 = -1/2 \)

and eigenvectors \( \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).

We can diagonalise with \( B = \begin{bmatrix} \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/2} & \sqrt{1/2} \end{bmatrix} = \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix} \).

Then \( B^t A B = B^t AB = \begin{bmatrix} \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/2} & \sqrt{1/2} \end{bmatrix} \) is diagonal.

If we change variables \( \vec{x} = B \vec{y} \)

we get \( \vec{y}^t A \vec{y} = (B \vec{y})^t A (B \vec{y}) = \vec{y}^t (B^t A B) \vec{y} \)

so we get a quadratic with diagonal entries.

In this case \( \vec{y}^t A \vec{y} = \frac{1}{2} y_1^2 - \frac{1}{2} y_2^2 \) with \( \vec{y} = B^t \vec{x} \).

In the new variables \( \vec{y} = B^t \vec{x} \)

\[
\begin{align*}
\quad xy &= \frac{1}{2} (x+y)^2 - \frac{1}{2} (x+y)^2 \\
\rightarrow y_1^2 - y_2^2 &= 2
\end{align*}
\]
More generally, a quadratic \( x^t A x \) with \( A \) symmetric can be expressed as \((B^t y)^t A (B^t y) = y^t (B^t A B) y = \sum_{i=1}^{n} \lambda_i y_i^2 \), where \( B \) is orthogonal and \( \lambda_i \) are the eigenvalues.

**Example.** Consider \( x^2 - y^2 + z^2 - 2xy - 2xz - 2yz = 1 \).

In this case, \( A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \). The eigenvalues are

\( \lambda_1 = 1 \), \( \lambda_2 = 2 \) and \( \lambda_3 = -2 \) with eigenvectors

\( v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), \( v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \) and \( v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \). We divide each of these by its length to get

\( B = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \Rightarrow B^t A B = B^{-1} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \).

This proves: we can change variables by \( \mathbf{X} = B \mathbf{Y} \) to write

\( x^2 - y^2 + z^2 - 2xy - 2xz - 2yz = \sum_{i=1}^{3} \lambda_i y_i^2 = y_1^2 + 2y_2^2 - 2y_3^2 \).

Thus, the original equation becomes \( y_1^2 + 2y_2^2 - 2y_3^2 = 1 \), an \( 1 \)-sided hyperbolid (that does not meet the \( y_3 \)-axis).

The new variables \( y_1, y_2, y_3 \) can be determined as

\( \mathbf{X} = B \mathbf{Y} \) or \( \mathbf{Y} = B^{-1} \mathbf{X} = B^t \mathbf{X} \),

or

\( \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \).

We have actually shown

\( x^2 - y^2 + z^2 - 2xy - 2xz - 2yz = y_1^2 + y_2^2 - 2y_3^2 
= \left( \frac{x+y-2z}{\sqrt{3}} \right)^2 + 2 \left( \frac{-x+y+z}{\sqrt{2}} \right)^2 - 2 \left( \frac{x+y+z}{\sqrt{6}} \right)^2 \).
We say $A$ is positive definite symmetric, if $A^t = A$ (symmetric) and $\langle x, x \rangle = x^t A x$ is positive for all $x \neq 0$.

**Theorem (Tests for positive definiteness)** Suppose $A$ is symmetric & real. Then $A$ is positive definite if and only if:

1. **Definition** ---- $x^t A x$ is positive for all $x \neq 0$.
2. **Eigenvalues** ---- $\lambda > 0$ for all eigenvalues $\lambda$ of $A$.
3. **Sylvester's criterion** ---- the determinants of all $k \times k$ upper left submatrices are all positive.

### Example 1.
Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$. To check it is pos. def., we note that $3 > 0$ and $\det A = 3 \cdot 4 - 1 > 0$. Alternatively, we can compute eigenvalues $\lambda^2 - 7\lambda + 11 = 0 \Rightarrow \lambda = \frac{7 \pm \sqrt{5}}{2}$ are both positive.

### Example 2.
Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 5 \\ 0 & 5 & x \end{bmatrix}$ with $x$ some parameter.

The eigenvalues are difficult to find explicitly. Using Sylvester's criterion ---- $3 > 0$ and $\det \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = 6 - 1 > 0$ so we need $\det A > 0$ as well.

In this case $\det A = 5x - 5 \cdot 15 = 5(x - 15)$ so $A$ is positive definite $\iff x > 15$.

**Proof:** 1 & 2 equivalent. We need $x^t A x$ positive $\forall x \neq 0$.

Let $x = B y$ with $B$ orthogonal and $B^t A B = B^t AB = [A_{ij}]$.

We get $x^t A x = y^t (B^t A B) y = \sum \lambda_i y_i^2$. This is positive $\forall y \neq 0$ if and only if $\lambda_i > 0 \forall i$. \(\blacksquare\)
Example 3. Let \( A = \begin{bmatrix} 2 & x \\ x & x+4 \\ 0 & x-4 \end{bmatrix} \). In this case,

(a) \( \det A \) is positive \( \sqrt{ } \)

(b) \( \det \begin{bmatrix} 2 & x \\ x & x+4 \end{bmatrix} = 2x+8 - x^2 \) should be positive.

Let's factor ... \( 2x+8 - x^2 = -(x^2 - 2x - 8) \)

\[ = -(x-4)(x+2) \]

\(-2 < x < 4\)

\( \begin{array}{c}
2 \\
\hline
4 \\
\hline
\end{array} \)

(c) \( \det A > 0 \)

\( \det A = -x^3 - x^2 + 26x - 24 \)

Roots are ... \( x = 1, 4, -6 \)

so \( \det A = -(x-1)(x-4)(x+6) \)

Conclusion: \( A \) pos. def. \( \Rightarrow 1 < x < 4 \).