1. Find an orthogonal matrix $B$ such that $B'AB$ is diagonal when

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$ 

The eigenvalues of $A$ are the roots of the characteristic polynomial

$$f(\lambda) = \lambda^2 - (\text{tr} A)\lambda + \det A = \lambda^2 - 9\lambda + 14 = (\lambda - 2)(\lambda - 7),$$

namely $\lambda_1 = 2$ and $\lambda_2 = 7$. It is easy to check that the corresponding eigenvectors are

$$v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ 

Dividing each of those by its length, one obtains an orthogonal matrix $B$ such that

$$B = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \Rightarrow B'AB = B^{-1}AB = \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$ 

2. A real matrix $A$ is called skew-symmetric, if $A' = -A$. Show that the eigenvalues of such a matrix are purely imaginary, namely of the form $\lambda = iy$ with $y \in \mathbb{R}$.

Assuming that $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, one finds that

$$\lambda \langle v, v \rangle = \langle v, \lambda v \rangle = \langle v, Av \rangle = \langle A'v, v \rangle.$$

Since $A^* = A'$, conjugate linearity in the first variable now gives

$$\lambda \langle v, v \rangle = \langle A^*v, v \rangle = \langle -Av, v \rangle = \langle -\lambda v, v \rangle = -\overline{\lambda} \langle v, v \rangle.$$

Writing $\lambda = x + iy$ for some real numbers $x$ and $y$, we conclude that

$$\lambda = -\overline{\lambda} \quad \Rightarrow \quad x + iy = -x + iy \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad \lambda = iy.$$ 

3. Suppose that $v_1, v_2, \ldots, v_n$ form an orthonormal basis of $\mathbb{R}^n$ and consider the $n \times n$ matrix $A = I_n - 2v_1v_1'$. Show that $A^2 = I_n$ and find the Jordan form of $A$.

When it comes to the first part, it is easy to check that

$$A^2 = (I_n - 2v_1v_1')(I_n - 2v_1v_1') = I_n - 4v_1v_1' + 4v_1(v_1'v_1)v_1' = I_n.$$ 

When it comes to the second part, the given vectors are all eigenvectors because

$$Av_1 = v_1 - 2v_1(v_1'v_1) = -v_1, \quad Av_k = v_k - 2v_1(v_1'v_k) = v_k$$

for each $k \geq 2$. This means that $A$ has $n$ linearly independent eigenvectors, so its Jordan form is diagonal and the diagonal entries are the eigenvalues $\lambda = -1, 1, 1, \ldots, 1$. 

4. Find a $2 \times 2$ symmetric matrix $A$ with eigenvalues $\lambda = 1, 2$ such that $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector of $A$ with eigenvalue $\lambda_1 = 1$.

The eigenvalues are distinct, so the eigenvectors are orthogonal to one another and the second eigenvector is a scalar multiple of $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$. Dividing each of the eigenvectors by its length, one obtains an orthogonal matrix $B$ such that

$$B = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \quad \implies \quad B^t A B = B^{-1} A B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ 

Once we now solve this equation for $A$, we may finally conclude that

$$A = \frac{1}{25} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 41 \\ -12 \end{bmatrix}.$$

5. Find an orthonormal basis of $\mathbb{R}^3$ that consists entirely of eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

The eigenvalues of $A$ are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 6\lambda^2 + \lambda^2(6 - \lambda),$$

namely $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 6$. The corresponding eigenvectors are easily found to be

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Note that the first two vectors are not orthogonal to one another. To find an orthogonal basis, we thus resort to the Gram-Schmidt procedure which gives the vectors

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\right) \mathbf{w}_1 = \begin{bmatrix} -1/5 \\ -2/5 \\ 1 \end{bmatrix}, \quad \mathbf{w}_3 = \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Once we now divide each of these vectors by its length, we get the orthonormal basis

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$