1. Find a basis for both the null space and the column space of the matrix

\[
A = \begin{bmatrix}
1 & 1 & 3 & 4 \\
2 & 0 & 2 & 6 \\
1 & 1 & 3 & 4
\end{bmatrix}.
\]

The reduced row echelon form of \(A\) is easily found to be

\[
R = \begin{bmatrix}
1 & 0 & 1 & 3 \\
0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Since the pivots appear in the first and the second columns, this implies that

\[
\mathcal{C}(A) = \text{Span}\{Ae_1, Ae_2\} = \text{Span}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.
\]

The null space of \(A\) is the same as the null space of \(R\). It can be expressed in the form

\[
\mathcal{N}(A) = \left\{ \begin{bmatrix} -x_3 - 3x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]

2. Find the eigenvalues and the generalised eigenvectors of the matrix

\[
A = \begin{bmatrix}
4 & -6 & 3 \\
0 & -1 & 4 \\
1 & -2 & 2
\end{bmatrix}.
\]

The eigenvalues of \(A\) are the roots of the characteristic polynomial

\[
f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = (3 - \lambda)(\lambda - 1)^2.
\]

When it comes to the eigenvalue \(\lambda = 3\), one can easily check that

\[
\mathcal{N}(A - 3I) = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 3I)^2 = \mathcal{N}(A - 3I).
\]
This implies that \( \mathcal{N}(A - 3I)^j = \mathcal{N}(A - 3I) \) for all \( j \geq 1 \), so we have found all generalised eigenvectors with \( \lambda = 3 \). When it comes to the eigenvalue \( \lambda = 1 \), one similarly has
\[
\mathcal{N}(A - I) = \text{Span} \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - I)^2 = \mathcal{N}(A - I)^3 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]
In view of the general theory, we must thus have \( \mathcal{N}(A - I)^j = \mathcal{N}(A - I)^2 \) for all \( j \geq 2 \).

3. Find a relation between \( x, y, z \) such that the following matrix is diagonalisable.

\[
A = \begin{bmatrix}
1 & x & y \\
0 & 2 & z \\
0 & 0 & 1
\end{bmatrix}.
\]

Since \( A \) is upper triangular, its eigenvalues are its diagonal entries \( \lambda = 1, 1, 2 \). Let us now worry about the eigenvectors. When \( \lambda = 2 \), we can use row reduction to get
\[
A - 2I = \begin{bmatrix}
-1 & x & y \\
0 & 0 & z \\
0 & 0 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -x & -y \\
0 & 0 & 1 \\
0 & 0 & z
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -x & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}.
\]
This gives 2 pivots and 1 linearly independent eigenvector. When \( \lambda = 1 \), we get
\[
A - I = \begin{bmatrix}
0 & x & y \\
0 & 1 & z \\
0 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 1 & z \\
0 & x & y \\
0 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 1 & z \\
0 & 0 & y - xz \\
0 & 0 & 0
\end{bmatrix}.
\]
This gives 2 pivots and 1 linearly independent eigenvector in the case that \( y \neq xz \), but it gives 1 pivot and 2 linearly independent eigenvectors in the case that \( y = xz \). Thus, \( A \) is diagonalisable if and only if \( y = xz \).

4. Suppose that \( \lambda \) is an eigenvalue of a square matrix \( A \) and let \( j \) be a positive integer. Show that the null space of \((A - \lambda I)^j\) is an \( A \)-invariant subspace of \( \mathbb{C}^n \).

Assuming that \( \mathbf{v} \) is in the null space of \((A - \lambda I)^j\), one can easily check that
\[
(A - \lambda I)^j \mathbf{v} = (A - \lambda I)^j (A - \lambda I + \lambda I) \mathbf{v} = (A - \lambda I)^{j+1} \mathbf{v} + \lambda (A - \lambda I)^j \mathbf{v} = 0.
\]
This means that \( A \mathbf{v} \) is also in the null space, so the null space is \( A \)-invariant.