1. Find the eigenvalues and the eigenvectors of the matrix

\[ A = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}. \]

Since \( \text{tr} \ A = 8 \) and \( \det A = 15 - 8 = 7 \), the characteristic polynomial is

\[ f(\lambda) = \lambda^2 - (\text{tr} \ A)\lambda + \det A = \lambda^2 - 8\lambda + 7 = (\lambda - 1)(\lambda - 7). \]

The eigenvectors with eigenvalue \( \lambda = 1 \) satisfy the system \( A\mathbf{v} = \mathbf{v} \), namely

\[ (A - I)\mathbf{v} = 0 \implies \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 4x + 2y = 0 \implies y = -2x. \]

This means that every eigenvector with eigenvalue \( \lambda = 1 \) must have the form

\[ \mathbf{v} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad x \neq 0. \]

Similarly, the eigenvectors with eigenvalue \( \lambda = 7 \) are solutions of \( A\mathbf{v} = 7\mathbf{v} \), so

\[ (A - 7I)\mathbf{v} = 0 \implies \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2x - 2y = 0 \implies y = x \]

and every eigenvector with eigenvalue \( \lambda = 7 \) must have the form

\[ \mathbf{v} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x \neq 0. \]

2. Is the following matrix diagonalisable? Why or why not?

\[ A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}. \]

Since \( \text{tr} \ A = 6 \) and \( \det A = 9 \), the characteristic polynomial of \( A \) is

\[ f(\lambda) = \lambda^2 - (\text{tr} \ A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2, \]

so the only eigenvalue is \( \lambda = 3 \). The eigenvectors satisfy the system \( (A - 3I)\mathbf{v} = 0 \), namely

\[ \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x + y = 0 \implies \mathbf{v} = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad x \neq 0. \]

Since \( A \) has only one linearly independent eigenvector, it is not diagonalisable.
3. Find a matrix $A$ that has $v_1$ as an eigenvector with eigenvalue $\lambda_1 = 2$ and $v_2$ as an eigenvector with eigenvalue $\lambda_2 = 5$ when

$$v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

If $B$ is the matrix whose columns are $v_1$ and $v_2$, then the general theory implies that

$$B^{-1}AB = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$ 

Once we now solve this equation for $A$, we may conclude that

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 4 & 5 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$ 

4. Two square matrices $A, C$ are said to be similar, if $C = B^{-1}AB$ for some invertible matrix $B$. Show that similar matrices have the same characteristic polynomial and also the same eigenvalues. Hint: one has $C - \lambda I = B^{-1}(A - \lambda I)B$.

Using the identity in the hint and properties of the determinant, we get

$$\det(C - \lambda I) = \det B^{-1} \cdot \det(A - \lambda I) \cdot \det B = \det(A - \lambda I).$$

This shows that $A, C$ have the same characteristic polynomial. The eigenvalues are merely the roots of this polynomial, so $A, C$ have the same eigenvalues as well.