1. Find the eigenvalues and the eigenvectors of the matrix

\[ A = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}. \]

Since \( \text{tr} A = 8 \) and \( \det A = 15 - 8 = 7 \), the characteristic polynomial is

\[ f(\lambda) = \lambda^2 - (\text{tr} A)\lambda + \det A = \lambda^2 - 8\lambda + 7 = (\lambda - 1)(\lambda - 7). \]

The eigenvectors with eigenvalue \( \lambda = 1 \) satisfy the system \( A\mathbf{v} = \mathbf{v} \), namely

\[ (A - I)\mathbf{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad 4x + 2y = 0 \quad \Rightarrow \quad y = -2x. \]

This means that every eigenvector with eigenvalue \( \lambda = 1 \) must have the form

\[ \mathbf{v} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad x \neq 0. \]

Similarly, the eigenvectors with eigenvalue \( \lambda = 7 \) are solutions of \( A\mathbf{v} = 7\mathbf{v} \), so

\[ (A - 7I)\mathbf{v} = 0 \quad \Rightarrow \quad \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad 2x - 2y = 0 \quad \Rightarrow \quad y = x \]

and every eigenvector with eigenvalue \( \lambda = 7 \) must have the form

\[ \mathbf{v} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x \neq 0. \]

2. Is the following matrix diagonalisable? Why or why not?

\[ A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}. \]

Since \( \text{tr} A = 6 \) and \( \det A = 9 \), the characteristic polynomial of \( A \) is

\[ f(\lambda) = \lambda^2 - (\text{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2, \]

so the only eigenvalue is \( \lambda = 3 \). The eigenvectors satisfy the system \( (A - 3I)\mathbf{v} = 0 \), namely

\[ \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad x + y = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad x \neq 0. \]

Since \( A \) has only one linearly independent eigenvector, it is not diagonalisable.
3. Find a matrix $A$ that has $v_1$ as an eigenvector with eigenvalue $\lambda_1 = 2$ and $v_2$ as an eigenvector with eigenvalue $\lambda_2 = 5$ when

$$v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

If $B$ is the matrix whose columns are $v_1$ and $v_2$, then the general theory implies that

$$B^{-1}AB = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$ 

Once we now solve this equation for $A$, we may conclude that

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 4 & 5 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$ 

4. Two square matrices $A, C$ are said to be similar, if $C = B^{-1}AB$ for some invertible matrix $B$. Show that similar matrices have the same characteristic polynomial and also the same eigenvalues. Hint: one has $C - \lambda I = B^{-1}(A - \lambda I)B$.

Using the identity in the hint and properties of the determinant, we get

$$\det(C - \lambda I) = \det B^{-1} \cdot \det(A - \lambda I) \cdot \det B = \det(A - \lambda I).$$

This shows that $A, C$ have the same characteristic polynomial. The eigenvalues are merely the roots of this polynomial, so $A, C$ have the same eigenvalues as well.
1. Find a basis for both the null space and the column space of the matrix

\[ A = \begin{bmatrix}
1 & 1 & 3 & 4 \\
2 & 0 & 2 & 6 \\
1 & 1 & 3 & 4
\end{bmatrix}. \]

The reduced row echelon form of \( A \) is easily found to be

\[ R = \begin{bmatrix}
1 & 0 & 1 & 3 \\
0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}. \]

Since the pivots appear in the first and the second columns, this implies that

\[ C(A) = \text{Span}\{Ae_1, Ae_2\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}. \]

The null space of \( A \) is the same as the null space of \( R \). It can be expressed in the form

\[ \mathcal{N}(A) = \text{Span}\left\{ \begin{bmatrix} -x_3 - 3x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}. \]

2. Find the eigenvalues and the generalised eigenvectors of the matrix

\[ A = \begin{bmatrix}
4 & -6 & 3 \\
0 & -1 & 4 \\
1 & -2 & 2
\end{bmatrix}. \]

The eigenvalues of \( A \) are the roots of the characteristic polynomial

\[ f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = (3 - \lambda)(\lambda - 1)^2. \]

When it comes to the eigenvalue \( \lambda = 3 \), one can easily check that

\[ \mathcal{N}(A - 3I) = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 3I)^2 = \mathcal{N}(A - 3I). \]
This implies that $\mathcal{N}(A - 3I)^j = \mathcal{N}(A - 3I)$ for all $j \geq 1$, so we have found all generalised eigenvectors with $\lambda = 3$. When it comes to the eigenvalue $\lambda = 1$, one similarly has

$$\mathcal{N}(A - I) = \text{Span} \begin{Bmatrix} 3 \\ 2 \\ 1 \end{Bmatrix}, \quad \mathcal{N}(A - I)^2 = \mathcal{N}(A - I)^3 = \text{Span} \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{Bmatrix}.$$

In view of the general theory, we must thus have $\mathcal{N}(A - I)^j = \mathcal{N}(A - I)^2$ for all $j \geq 2$.

3. Find a relation between $x, y, z$ such that the following matrix is diagonalisable.

$$A = \begin{bmatrix} 1 & x & y \\ 0 & 2 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $A$ is upper triangular, its eigenvalues are its diagonal entries $\lambda = 1, 1, 2$. Let us now worry about the eigenvectors. When $\lambda = 2$, we can use row reduction to get

$$A - 2I = \begin{bmatrix} -1 & x & y \\ 0 & 0 & z \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -x & -y \\ 0 & 0 & 1 \\ 0 & 0 & z \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -x & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives 2 pivots and 1 linearly independent eigenvector. When $\lambda = 1$, we get

$$A - I = \begin{bmatrix} 0 & x & y \\ 0 & 1 & z \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & z \\ 0 & x & y \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & z \\ 0 & 0 & y - xz \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives 2 pivots and 1 linearly independent eigenvector in the case that $y \neq xz$, but it gives 1 pivot and 2 linearly independent eigenvectors in the case that $y = xz$. Thus, $A$ is diagonalisable if and only if $y = xz$.

4. Suppose that $\lambda$ is an eigenvalue of a square matrix $A$ and let $j$ be a positive integer. Show that the null space of $(A - \lambda I)^j$ is an $A$-invariant subspace of $\mathbb{C}^n$.

Assuming that $v$ is in the null space of $(A - \lambda I)^j$, one can easily check that

$$(A - \lambda I)^j A v = (A - \lambda I)^j (A - \lambda I + \lambda I) v = (A - \lambda I)^{j+1} v + \lambda (A - \lambda I)^j v = 0.$$ 

This means that $Av$ is also in the null space, so the null space is $A$-invariant.
1. Find the Jordan form and a Jordan basis for the matrix

\[ A = \begin{bmatrix} -1 & 1 & 2 \\ -7 & 5 & 3 \\ -5 & 1 & 6 \end{bmatrix}. \]

The characteristic polynomial of the given matrix is

\[ f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 10\lambda^2 - 33\lambda + 36 = (4 - \lambda)(\lambda - 3)^2. \]

Thus, the eigenvalues are \( \lambda = 3, 3, 4 \) and one can easily find the null spaces

\[ \mathcal{N}(A - 3I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 4I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}. \]

This implies that \( A \) is not diagonalisable and that its Jordan form is

\[ J = B^{-1}AB = \begin{bmatrix} 3 & \ast \\ 1 & 3 \end{bmatrix}. \]

To find a Jordan basis, we need to find a Jordan chain \( v_1, v_2 \) with eigenvalue \( \lambda = 3 \) and an eigenvector \( v_3 \) with eigenvalue \( \lambda = 4 \). In our case, we have

\[ \mathcal{N}(A - 3I)^2 = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \]

so it easily follows that a Jordan basis is provided by the vectors

\[ v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = (A - 3I)v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}. \]

2. Suppose that \( A \) is a \( 4 \times 4 \) matrix whose column space is equal to its null space. Show that \( A^2 \) must be the zero matrix and find the Jordan form of \( A \).

To show that \( A^2 \) is the zero matrix, we note that its columns are all zero, as

\[ Ae_i \in \mathcal{C}(A) \implies Ae_i \in \mathcal{N}(A) \implies A(Ae_i) = 0 \implies A^2e_i = 0. \]
To find the Jordan form of $A$, we note that the null space of $A$ is two-dimensional since
\[
\dim C(A) + \dim N(A) = 4 \implies 2 \dim N(A) = 4 \implies \dim N(A) = 2.
\]
Also, the null space of $A^2$ is four-dimensional because $A^2$ is the zero matrix by above. This gives the Jordan chain diagram
\[
\bullet \bullet
\]
for the eigenvalue $\lambda = 0$, so the Jordan form is
\[
J = \begin{bmatrix}
0 & 0 \\
1 & 0 \\
0 & 1 \\
1 & 0 \\
\end{bmatrix}.
\]

3. Suppose that $A$ is a $4 \times 4$ matrix whose only eigenvalue is $\lambda = 1$. Suppose also that the column space of $A - I$ is one-dimensional. Find the Jordan form of $A$.

The eigenvalue $\lambda = 1$ has multiplicity 4 and the number of Jordan chains is
\[
\dim N(A - I) = 4 - \dim C(A - I) = 3.
\]
In particular, the Jordan chain diagram is
\[
\bullet \bullet \bullet
\]
and the Jordan form is
\[
J = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{bmatrix}.
\]

4. Suppose that $v_1, v_2, \ldots, v_k$ form a basis for the null space of a square matrix $A$ and that $C = B^{-1}AB$ for some invertible matrix $B$. Find a basis for the null space of $C$.

We note that a vector $v$ lies in the null space of $C = B^{-1}AB$ if and only if
\[
B^{-1}ABv = 0 \iff ABv = 0 \iff Bv \in N(A) \iff Bv = \sum_{i=1}^k c_i v_i
\]
for some scalar coefficients $c_i$. In particular, the vectors $B^{-1}v_i$ span the null space of $C$. To show that these vectors are also linearly independent, we note that
\[
\sum_{i=1}^k c_i B^{-1}v_i = 0 \iff \sum_{i=1}^k c_i v_i = 0 \iff c_i = 0 \text{ for all } i.
\]