Linear algebra II
Homework #8 solutions

1. Find an orthogonal matrix B such that $B^t AB$ is diagonal when

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = -(\lambda - 1)(\lambda - 3)(\lambda - 6).$$

This gives $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = 6$, while the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since the eigenvalues are distinct, the eigenvectors are orthogonal to one another. Once we now divide each of them by its length, we obtain the columns of the orthogonal matrix

$$B = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

2. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form an orthonormal basis of $\mathbb{R}^n$ and consider the $n \times n$ matrix $A = \boldsymbol{v}_2 \boldsymbol{v}_1^t$. Show that $A^2 = 0$ and find the Jordan form of A.

Since $\boldsymbol{v}_1$ is perpendicular to $\boldsymbol{v}_2$ by assumption, one easily finds that

$$\boldsymbol{v}_1^t \boldsymbol{v}_2 = 0 \quad \Longrightarrow \quad A^2 = \boldsymbol{v}_2 \boldsymbol{v}_1^t \boldsymbol{v}_2 \boldsymbol{v}_1^t = 0.$$

In particular, $\lambda = 0$ is the only eigenvalue of A and we also have

$$A\boldsymbol{v}_1 = \boldsymbol{v}_2 \boldsymbol{v}_1^t \boldsymbol{v}_1 = \boldsymbol{v}_2, \qquad A\boldsymbol{v}_k = \boldsymbol{v}_2 \boldsymbol{v}_1^t \boldsymbol{v}_k = 0$$

for each integer $k \geq 2$. Since $A\boldsymbol{v}_1 = \boldsymbol{v}_2$ and $A\boldsymbol{v}_2 = 0$, the first two vectors form a Jordan chain of length 2. Thus, the Jordan form contains a 2×2 block with eigenvalue $\lambda = 0$. The remaining blocks correspond to the vectors $\boldsymbol{v}_3, \ldots, \boldsymbol{v}_n$ and they are all 1×1 blocks.

3. Find an orthonormal basis of $\mathbb{R}^3$ that consists entirely of eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 14\lambda^2 = \lambda^2(14 - \lambda),$$

namely $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 14$. The corresponding eigenvectors are easily found to be

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Note that the first two vectors are not orthogonal to one another. To find an orthogonal basis, we thus resort to the Gram-Schmidt procedure which gives the vectors

$$\boldsymbol{w}_1 = \boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 = \begin{bmatrix} -3/5 \\ -6/5 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Once we now divide each of these vectors by its length, we get the orthonormal basis

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \frac{1}{\sqrt{70}} \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}, \qquad \boldsymbol{u}_3 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

4. Find an orthogonal 3×3 matrix whose first two columns are

$$\boldsymbol{v}_1 = \begin{bmatrix} \cos x \cdot \cos y \\ \sin x \\ \cos x \cdot \sin y \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -\sin y \\ 0 \\ \cos y \end{bmatrix}, \qquad x, y \in \mathbb{R}.$$

It is easy to check that $\boldsymbol{v}_1, \boldsymbol{v}_2$ are orthogonal vectors of unit length, namely

$$\begin{aligned} \boldsymbol{v}_1^t \boldsymbol{v}_2 &= -\cos x \cdot \cos y \cdot \sin y + \cos x \cdot \cos y \cdot \sin y = 0, \\ ||\boldsymbol{v}_1||^2 &= \cos^2 x \cdot \cos^2 y + \cos^2 x \cdot \sin^2 y + \sin^2 x = \cos^2 x + \sin^2 x = 1, \\ ||\boldsymbol{v}_2||^2 &= \sin^2 y + \cos^2 y = 1. \end{aligned}$$

The third column is perpendicular to each of the first two columns, so it is parallel to

$$\boldsymbol{v}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{bmatrix} \cos x \cdot \cos y \\ \sin x \\ \cos x \cdot \sin y \end{bmatrix} \times \begin{bmatrix} -\sin y \\ 0 \\ \cos y \end{bmatrix} = \begin{bmatrix} \sin x \cdot \cos y \\ -\cos x \\ \sin x \cdot \sin y \end{bmatrix}.$$

This is actually a unit vector itself, so the third column could be either $\boldsymbol{v}_3$ or $-\boldsymbol{v}_3$.