1. Consider $\mathbb{R}^3$ with the usual dot product. Use the Gram-Schmidt procedure to find an orthogonal basis, starting with the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$ 

Keep the first vector and let $w_1 = v_1$. The second vector $v_2$ must be replaced by

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1 + 0 + 3}{1 + 4 + 9} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/7 \\ -4/7 \\ 1/7 \end{bmatrix}$$

and then the third vector $v_3$ must be replaced by

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ -1/3 \end{bmatrix}.$$ 

2. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle x, y \rangle = x_1 y_1 + x_1 y_2 + x_2 y_1 + 5 x_2 y_2.$$ 

Show that this is an inner product and use the Gram-Schmidt procedure to find an orthogonal basis for it, starting with the standard basis of $\mathbb{R}^2$.

The given form is symmetric because its matrix with respect to the standard basis is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}.$$ 

To show that the form is positive definite, we complete the square to write

$$\langle x, x \rangle = x_1^2 + 2x_1x_2 + 5x_2^2 = (x_1 + x_2)^2 + 4x_2^2.$$ 

Finally, one can obtain an orthogonal basis by taking $w_1 = e_1$ and

$$w_2 = e_2 - \frac{\langle e_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = e_2 - \frac{e_2^t A e_1}{e_1^t A e_1} e_1 = e_2 - \frac{a_{21}}{a_{11}} e_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$
3. Define a bilinear form on the space $M_{22}$ of all $2 \times 2$ real matrices by setting

$$\langle A, B \rangle = \text{tr}(A^tB)$$

for all $2 \times 2$ real matrices $A, B$. Express this equation in terms of the entries of the two matrices. Is the bilinear form symmetric? Is it positive definite?

In terms of the entries of the two matrices, we have

$$A^tB = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{21} & a_{11}b_{12} + a_{21}b_{22} \\ a_{12}b_{11} + a_{22}b_{21} & a_{12}b_{12} + a_{22}b_{22} \end{bmatrix},$$

so one may write the given equation as

$$\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22} = \sum_{i,j} a_{ij}b_{ij}.$$

In particular, the given form is symmetric and we also have

$$\langle A, A \rangle = a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2 = \sum_{i,j} a_{ij}^2 \geq 0.$$

Since equality holds if and only if $A = 0$, the given form is positive definite as well.

4. Find two eigenvectors of $A$ which form an orthonormal basis of $\mathbb{R}^2$ when

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad a, b \neq 0.$$

First of all, we compute the eigenvalues of $A$. The characteristic polynomial is

$$f(\lambda) = \lambda^2 - (\text{tr } A)\lambda + \det A = \lambda^2 - 2a\lambda + a^2 - b^2 = (\lambda - a)^2 - b^2,$$

so it easily follows that the eigenvalues are

$$(\lambda - a)^2 = b^2 \quad \implies \quad \lambda - a = \pm b \quad \implies \quad \lambda = a \pm b.$$

Next, we turn to the eigenvectors. When $\lambda = a + b$, one may use row reduction to get

$$A - \lambda I = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{N}(A - \lambda I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

When $\lambda = a - b$, one may use a similar computation to find that

$$A - \lambda I = \begin{bmatrix} b & b \\ b & b \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{N}(A - \lambda I) = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

This gives two vectors which are orthogonal to one another, so an orthonormal basis is

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$