1. Let \( x_0 = 1 \) and \( y_0 = 2 \). Suppose the sequences \( x_n, y_n \) are such that
\[
\begin{align*}
x_n &= 8x_{n-1} - 9y_{n-1}, \\
y_n &= x_{n-1} + 2y_{n-1}
\end{align*}
\]
for each \( n \geq 1 \). Determine each of \( x_n \) and \( y_n \) explicitly in terms of \( n \).

As we already know, one may express this problem in terms of matrices by writing
\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 8 & -9 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = A \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} \implies \begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.
\]

Let us now focus on computing \( A^n \). The characteristic polynomial of \( A \) is
\[
f(\lambda) = \lambda^2 - (\text{tr} \, A)\lambda + \det A = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2,
\]
so the only eigenvalue is \( \lambda = 5 \). The eigenvectors of \( A \) are nonzero elements of the null space
\[
N(A - 5I) = \text{Span} \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\},
\]
while \( (A - 5I)^2 \) is the zero matrix. Thus, a Jordan basis is provided by the vectors
\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} A - 5I \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \end{bmatrix}.
\]

Letting \( B \) be the matrix whose columns are \( v_1 \) and \( v_2 \), we must thus have
\[
J = B^{-1}AB = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \implies J^n = B^{-1}A^n B = \begin{bmatrix} 5^n \\ 5^{n-1} \end{bmatrix}.
\]

Once we now solve this equation for \( A^n \), we find that
\[
A^n = BJ^nB^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5^n \\ n5^{n-1} \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (5 + 3n)5^{n-1} \\ n5^{n-1} \end{bmatrix}.
\]

In particular, the sequences \( x_n, y_n \) are given explicitly by
\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} (1 - 3n)5^n \\ (2 - n)5^n \end{bmatrix}.
\]
2. Which of the following matrices are similar? Explain.

\[
A = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & -1 \\
1 & 3
\end{bmatrix}, \quad C = \begin{bmatrix}
2 & 0 \\
1 & 2
\end{bmatrix}, \quad D = \begin{bmatrix}
1 & 0 \\
1 & 2
\end{bmatrix}.
\]

The eigenvalues of the first matrix are given by

\[
\lambda^2 - (\text{tr} \ A)\lambda + \det A = 0 \implies \lambda^2 - 4\lambda + 3 = 0 \implies \lambda = 1, 3
\]

and the eigenvalues of the second matrix are

\[
\lambda^2 - (\text{tr} \ B)\lambda + \det B = 0 \implies \lambda^2 - 4\lambda + 4 = 0 \implies \lambda = 2, 2.
\]

The other two matrices are lower triangular, so their eigenvalues are their diagonal entries. This means that \(B, C\) are the only two matrices which could be similar. In fact, \(N(B - 2I) = \text{Span}\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}\) is one-dimensional, so the Jordan form of \(B\) has a single block and \(B\) is similar to \(C\).

3. Show that the trace of a square matrix \(A\) is the sum of its eigenvalues. Hint: prove the same statement for the Jordan form of \(A\) and then use similarity.

Let \(J\) denote the Jordan form of \(A\). Since \(J\) is lower triangular, its eigenvalues are its diagonal entries, so the sum of its eigenvalues is equal to its trace. On the other hand, \(A\) is similar to \(J\), so the two matrices have both the same trace and the same eigenvalues. This means that the sum of the eigenvalues of \(A\) is equal to the trace of \(A\).

4. Let \(x \in \mathbb{R}^3\) be nonzero and let \(A\) be the matrix whose columns are \(x, 2x, 3x\) in this order. Find the Jordan form of \(A\). Hint: the answer depends on the trace of \(A\); show that the null space is two-dimensional and that the eigenvalues are \(\lambda = 0, 0, \text{tr} \ A\).

Since the column space is one-dimensional, the null space must be two-dimensional. This means that the Jordan form contains two blocks with eigenvalue \(\lambda = 0\). As the sum of the eigenvalues is equal to the trace, the third eigenvalue is thus \(\lambda = \text{tr} \ A\).

Let us now consider two cases. When \(\text{tr} \ A \neq 0\), the eigenvalue \(\lambda = \text{tr} \ A\) is simple and it contributes a single \(1 \times 1\) block. There are also two Jordan blocks with \(\lambda = 0\), so all blocks are \(1 \times 1\) blocks and the Jordan form is diagonal. When \(\text{tr} \ A = 0\), the eigenvalue \(\lambda = 0\) is a triple eigenvalue that only contributes two Jordan blocks, so the Jordan form is

\[
J = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]