1. Let \( x_0 = 3 \) and \( y_0 = 1 \). Suppose the sequences \( x_n, y_n \) are such that
\[
x_n = 3x_{n-1} - 2y_{n-1}, \quad y_n = 4x_{n-1} + 9y_{n-1}
\]
for each \( n \geq 1 \). Determine each of \( x_n \) and \( y_n \) explicitly in terms of \( n \).

Letting \( u_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} \) and \( A = \begin{bmatrix} 3 & -2 \\ 4 & 9 \end{bmatrix} \), one easily finds that
\[
 u_n = Au_{n-1} = A^2u_{n-2} = \ldots = A^n u_0.
\]
In particular, it remains to compute \( A^n \). The eigenvalues of \( A \) are given by
\[
 \lambda^2 - (\text{tr } A)\lambda + \det A = 0 \quad \implies \quad \lambda^2 - 12\lambda + 35 = 0 \quad \implies \quad \lambda = 5, 7
\]
and we may proceed as usual to obtain the corresponding eigenvectors
\[
 v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.
\]
Let \( B \) be the matrix whose columns are \( v_1 \) and \( v_2 \). Then \( B^{-1}AB \) is diagonal and
\[
 B^{-1}AB = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \quad \implies \quad B^{-1}A^nB = \begin{bmatrix} 5^n & 0 \\ 0 & 7^n \end{bmatrix}.
\]
Solving this equation for \( A^n \) and simplifying, we now get
\[
 A^n = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^n & 0 \\ 0 & 7^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5^n - 7^n \quad 5^n - 7^n \\ 2 \cdot 7^n - 2 \cdot 5^n \quad 2 \cdot 7^n - 7^n \end{bmatrix}.
\]
In particular, the sequences \( x_n, y_n \) are given explicitly by
\[
 \begin{bmatrix} x_n \\ y_n \end{bmatrix} = u_n = A^n u_0 = \begin{bmatrix} 7 \cdot 5^n - 4 \cdot 7^n \\ 8 \cdot 7^n - 7 \cdot 5^n \end{bmatrix}.
\]

2. Show that the following matrix is diagonalisable.
\[
 A = \begin{bmatrix} 7 & 1 & -7 \\ 3 & 3 & -5 \\ 3 & 1 & -3 \end{bmatrix}.
\]

The eigenvalues of \( A \) are the roots of the characteristic polynomial
\[
 f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 14\lambda + 8 = -(\lambda - 1)(\lambda - 2)(\lambda - 4).
\]
Since the eigenvalues of \( A \) are distinct, we conclude that \( A \) is diagonalisable.
3. Find the eigenvalues and the generalised eigenvectors of the matrix

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
1 & 3 & -1 \\
0 & 1 & 2
\end{bmatrix}.
\]

The eigenvalues of \( A \) are the roots of the characteristic polynomial

\[
f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = (3 - \lambda)(\lambda - 2)^2.
\]

When it comes to the eigenvalue \( \lambda = 3 \), one can easily check that

\[
\mathcal{N}(A - 3I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 3I)^2 = \mathcal{N}(A - 3I).
\]

This implies that \( \mathcal{N}(A - 3I)^j = \mathcal{N}(A - 3I) \) for all \( j \geq 1 \), so we have found all generalised eigenvectors with \( \lambda = 3 \). When it comes to the eigenvalue \( \lambda = 2 \), one similarly has

\[
\mathcal{N}(A - 2I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 2I)^2 = \mathcal{N}(A - 2I)^3 = \mathcal{N}(A - 2I)^2.
\]

In view of the general theory, we must thus have \( \mathcal{N}(A - 2I)^j = \mathcal{N}(A - 2I)^2 \) for all \( j \geq 2 \).

4. Suppose that \( A \) is a \( 4 \times 4 \) matrix whose first two columns are linearly independent, its third column is equal to the first column and its last column is zero. Find a basis for both the column space and the null space of \( A \). Hint: one has \( Ae_3 = Ae_1 \) and \( Ae_4 = 0 \).

By assumption, the first two columns are linearly independent, while the other two columns are linear combinations of the first two. This implies that \( Ae_1, Ae_2 \) form a basis for the column space. Since the column space is two-dimensional, the null space must be two-dimensional as well. On the other hand, the given assumptions ensure that

\[
A(e_3 - e_1) = Ae_3 - Ae_1 = 0, \quad Ae_4 = 0.
\]

It easily follows that the vectors \( e_3 - e_1 \) and \( e_4 \) form a basis for the null space, namely

\[
\mathcal{N}(A) = \text{Span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]