1. Find the eigenvalues and the eigenvectors of the matrix

\[ A = \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}. \]

Since \( \text{tr} \ A = 9 \) and \( \det A = 18 + 2 = 20 \), the characteristic polynomial is

\[ f(\lambda) = \lambda^2 - (\text{tr} \ A)\lambda + \det A = \lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5). \]

The eigenvectors with eigenvalue \( \lambda = 4 \) satisfy the system \( A\mathbf{v} = 4\mathbf{v} \), namely

\[ (A - 4I)\mathbf{v} = 0 \implies \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -x + 2y = 0 \implies x = 2y. \]

This means that every eigenvector with eigenvalue \( \lambda = 4 \) must have the form

\[ \mathbf{v} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad y \neq 0. \]

Similarly, the eigenvectors with eigenvalue \( \lambda = 5 \) are solutions of \( A\mathbf{v} = 5\mathbf{v} \), so

\[ (A - 5I)\mathbf{v} = 0 \implies \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -x + y = 0 \implies x = y \]

and every eigenvector with eigenvalue \( \lambda = 5 \) must have the form

\[ \mathbf{v} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad y \neq 0. \]

2. Find the eigenvalues and the eigenvectors of the matrix

\[ A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 2 & 2 & -1 \end{bmatrix}. \]

The eigenvalues of \( A \) are the roots of the characteristic polynomial

\[ f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = -(\lambda - 1)^2(\lambda - 3). \]

The eigenvectors of \( A \) are nonzero vectors in the null spaces

\[ \mathcal{N}(A - I) = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 3I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}. \]
3. The following matrix has eigenvalues $\lambda = 1, 1, 2, 2$. Is it diagonalisable? Explain.

$$A = \begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 2 & -1 & 1 \\
-1 & 1 & 0 & 2 \\
-2 & 2 & -1 & 3
\end{bmatrix}.$$ 

When it comes to the eigenvalue $\lambda = 1$, row reduction of $A - \lambda I$ gives

$$A - \lambda I = \begin{bmatrix}
0 & 1 & -1 & 1 \\
0 & 1 & -1 & 1 \\
-1 & 1 & -1 & 2 \\
-2 & 2 & -1 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 \\
1 & -1 & 1 & -2 \\
0 & 0 & 1 & -2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

so there are 3 pivots and only 1 linearly independent eigenvector. When $\lambda = 2$, we have

$$A - \lambda I = \begin{bmatrix}
-1 & 1 & -1 & 1 \\
0 & 0 & -1 & 1 \\
-1 & 1 & -2 & 2 \\
-2 & 2 & -1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
-1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
-1 & 1 & 0 & 0 \\
-2 & 2 & 0 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},$$

so we get 2 pivots and 2 linearly independent eigenvectors. This gives a total of 3 linearly independent eigenvectors, so $A$ is not diagonalisable. One does not really need to find the eigenvectors in this case, but those are nonzero elements of the null spaces

$$\mathcal{N}(A - I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 2I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$ 

4. Suppose $A$ is a diagonalisable matrix and let $k \geq 1$ be an integer. Show that each eigenvector of $A$ is an eigenvector of $A^k$ and conclude that $A^k$ is diagonalisable.

If $v$ is an eigenvector of $A$ with eigenvalue $\lambda$, then $v$ satisfies $Av = \lambda v$, so

$$A^2v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v.$$ 

It easily follows by induction that $A^k v = \lambda^k v$ for each $k$. In particular, $v$ is an eigenvector of $A^k$ as well. Suppose that $A$ is $n \times n$. Being diagonalisable, it must then have $n$ linearly independent eigenvectors. Those are also eigenvectors of $A^k$, so this matrix has $n$ linearly independent eigenvectors and it is diagonalisable as well.
1. Let $x_0 = 3$ and $y_0 = 1$. Suppose the sequences $x_n, y_n$ are such that
\[ x_n = 3x_{n-1} - 2y_{n-1}, \quad y_n = 4x_{n-1} + 9y_{n-1} \]
for each $n \geq 1$. Determine each of $x_n$ and $y_n$ explicitly in terms of $n$.

Letting $u_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -2 \\ 4 & 9 \end{bmatrix}$, one easily finds that
\[ u_n = A u_{n-1} = A^2 u_{n-2} = \ldots = A^n u_0. \]

In particular, it remains to compute $A^n$. The eigenvalues of $A$ are given by
\[ \lambda^2 - (\text{tr} A) \lambda + \det A = 0 \quad \Rightarrow \quad \lambda^2 - 12 \lambda + 35 = 0 \quad \Rightarrow \quad \lambda = 5, 7 \]
and we may proceed as usual to obtain the corresponding eigenvectors
\[ v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \]

Let $B$ be the matrix whose columns are $v_1$ and $v_2$. Then $B^{-1} A B$ is diagonal and
\[ B^{-1} A B = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \quad \Rightarrow \quad B^{-1} A^n B = \begin{bmatrix} 5^n & 0 \\ 0 & 7^n \end{bmatrix}. \]

Solving this equation for $A^n$ and simplifying, we now get
\[ A^n = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^n & 0 \\ 0 & 7^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5^n - 7^n & 5^n - 7^n \\ 2 \cdot 7^n - 2 \cdot 5^n & 2 \cdot 7^n - 5^n \end{bmatrix}. \]

In particular, the sequences $x_n, y_n$ are given explicitly by
\[ \begin{bmatrix} x_n \\ y_n \end{bmatrix} = u_n = A^n u_0 = \begin{bmatrix} 7 \cdot 5^n - 4 \cdot 7^n \\ 8 \cdot 7^n - 7 \cdot 5^n \end{bmatrix}. \]

2. Show that the following matrix is diagonalisable.
\[ A = \begin{bmatrix} 7 & 1 & -7 \\ 3 & 3 & -5 \\ 3 & 1 & -3 \end{bmatrix}. \]

The eigenvalues of $A$ are the roots of the characteristic polynomial
\[ f(\lambda) = \det( A - \lambda I) = -\lambda^3 + 7\lambda^2 - 14\lambda + 8 = -(\lambda - 1)(\lambda - 2)(\lambda - 4). \]
Since the eigenvalues of $A$ are distinct, we conclude that $A$ is diagonalisable.
3. Find the eigenvalues and the generalised eigenvectors of the matrix

\[ A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}. \]

The eigenvalues of \( A \) are the roots of the characteristic polynomial

\[ f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = (3 - \lambda)(\lambda - 2)^2. \]

When it comes to the eigenvalue \( \lambda = 3 \), one can easily check that

\[ \mathcal{N}(A - 3I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 3I)^2 = \mathcal{N}(A - 3I). \]

This implies that \( \mathcal{N}(A - 3I)^j = \mathcal{N}(A - 3I) \) for all \( j \geq 1 \), so we have found all generalised eigenvectors with \( \lambda = 3 \). When it comes to the eigenvalue \( \lambda = 2 \), one similarly has

\[ \mathcal{N}(A - 2I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 2I)^2 = \mathcal{N}(A - 2I)^3 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}. \]

In view of the general theory, we must thus have \( \mathcal{N}(A - 2I)^j = \mathcal{N}(A - 2I)^2 \) for all \( j \geq 2 \).

4. Suppose that \( A \) is a \( 4 \times 4 \) matrix whose first two columns are linearly independent, its third column is equal to the first column and its last column is zero. Find a basis for both the column space and the null space of \( A \). Hint: one has \( A\mathbf{e}_3 = A\mathbf{e}_1 \) and \( A\mathbf{e}_4 = 0 \).

By assumption, the first two columns are linearly independent, while the other two columns are linear combinations of the first two. This implies that \( A\mathbf{e}_1, A\mathbf{e}_2 \) form a basis for the column space. Since the column space is two-dimensional, the null space must be two-dimensional as well. On the other hand, the given assumptions ensure that

\[ A(e_3 - e_1) = A\mathbf{e}_3 - A\mathbf{e}_1 = 0, \quad A\mathbf{e}_4 = 0. \]

It easily follows that the vectors \( e_3 - e_1 \) and \( e_4 \) form a basis for the null space, namely

\[ \mathcal{N}(A) = \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}. \]
1. The following matrix has $\lambda = 2$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix}
4 & -1 & -1 \\
2 & 1 & -1 \\
2 & -1 & 1
\end{bmatrix}.$$

In this case, the null space of $A - 2I$ is two-dimensional, as row reduction gives

$$A - 2I = \begin{bmatrix}
2 & -1 & -1 \\
2 & -1 & -1 \\
2 & -1 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1/2 & -1/2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.$$

On the other hand, $(A - 2I)^2$ is the zero matrix, so its null space is three-dimensional. Thus, the diagram of Jordan chains is $\bullet \bullet$ and there is a Jordan chain of length 2 as well as a Jordan chain of length 1. These Jordan chains give a $2 \times 2$ block and an $1 \times 1$ block, so

$$J = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}.$$

2. The following matrix has $\lambda = 2$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix}
3 & 2 & -1 \\
2 & 2 & -1 \\
-1 & 6 & 1
\end{bmatrix}.$$

In this case, the null space of $A - 2I$ is one-dimensional, as row reduction gives

$$A - 2I = \begin{bmatrix}
1 & 2 & -1 \\
2 & 0 & -1 \\
-1 & 6 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & -1 \\
0 & -4 & 1 \\
0 & 8 & -2
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -1/2 \\
0 & 1 & -1/4 \\
0 & 0 & 0
\end{bmatrix}.$$

Similarly, the null space of $(A - 2I)^2$ is two-dimensional because

$$(A - 2I)^2 = \begin{bmatrix}
6 & -4 & -2 \\
3 & -2 & -1 \\
12 & -8 & -4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -2/3 & -1/3 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},$$

while $(A - 2I)^3$ is the zero matrix, so its null space is three-dimensional. The diagram of Jordan chains is then $\bullet$ and there is a single $3 \times 3$ Jordan block, namely

$$J = \begin{bmatrix}
\lambda & 1 & 0 \\
1 & \lambda & 0 \\
0 & 0 & \lambda
\end{bmatrix} = \begin{bmatrix}
2 & 1 & 2 \\
1 & 2 & 1 \\
2 & 1 & 2
\end{bmatrix}.$$
3. Find a Jordan chain of length 2 for the matrix
\[ A = \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}. \]

The eigenvalues of the given matrix are the roots of the characteristic polynomial
\[ f(\lambda) = \lambda^2 - (\text{tr} \ A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2, \]
so \( \lambda = 3 \) is the only eigenvalue. Using row reduction, we now get
\[ A - 3I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \]
so the null space of \( A - 3I \) is one-dimensional. On the other hand, \( (A - 3I)^2 \) is the zero matrix, so its null space is two-dimensional. To find a Jordan chain of length 2, we pick a vector \( \mathbf{v}_1 \) that lies in the latter null space, but not in the former. We can always take \( \mathbf{v}_1 = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = (A - 3I)\mathbf{v}_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}, \]
but there are obviously infinitely many choices. Another possible choice would be \( \mathbf{v}_1 = \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \mathbf{v}_2 = (A - 3I)\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}. \)

4. Let \( \mathbf{x} \in \mathbb{R}^3 \) be nonzero and let \( A \) be the matrix whose columns are \( \mathbf{x}, 2\mathbf{x}, 3\mathbf{x} \) in this order. Show that \( \mathbf{x} \) is an eigenvector of \( A \) and find a basis for the null space of \( A \).

The columns of \( A \) are \( A\mathbf{e}_1 = \mathbf{x}, A\mathbf{e}_2 = 2\mathbf{x} \) and \( A\mathbf{e}_3 = 3\mathbf{x} \). It easily follows that
\[ A\mathbf{x} = A(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3) = x_1\mathbf{x} + x_2(2\mathbf{x}) + x_3(3\mathbf{x}) = \lambda\mathbf{x}, \]
where \( \lambda = x_1 + 2x_2 + 3x_3 \). This shows that \( \mathbf{x} \) is an eigenvector of \( A \). Since every column of \( A \) is a scalar multiple of \( \mathbf{x} \), the column space is one-dimensional and the null space is two-dimensional. Using the condition \( 2A\mathbf{e}_1 = 2\mathbf{x} = A\mathbf{e}_2 \), one finds that \( 2\mathbf{e}_1 - \mathbf{e}_2 \in \mathcal{N}(A) \).
Using the condition \( 3A\mathbf{e}_1 = 3\mathbf{x} = A\mathbf{e}_3 \), we get \( 3\mathbf{e}_1 - \mathbf{e}_3 \in \mathcal{N}(A) \) as well, hence
\[ \mathcal{N}(A) = \text{Span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}. \]
1. Find the Jordan form and a Jordan basis for the matrix

\[
A = \begin{bmatrix}
3 & 4 & -2 \\
2 & 5 & -2 \\
4 & 8 & -3
\end{bmatrix}.
\]

The characteristic polynomial of the given matrix is

\[
f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = (3 - \lambda)(\lambda - 1)^2,
\]

so its eigenvalues are \(\lambda = 1, 1, 3\). The corresponding null spaces are easily found to be

\[
\mathcal{N}(A - I) = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 3I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.
\]

These contain 3 linearly independent eigenvectors, so \(A\) is diagonalisable and

\[
B = \begin{bmatrix}
-2 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 2
\end{bmatrix} \implies J = B^{-1}AB = \begin{bmatrix} 1 & 0 \\
0 & 3 \end{bmatrix}.
\]

2. Find the Jordan form and a Jordan basis for the matrix

\[
A = \begin{bmatrix}
3 & 2 & -1 \\
1 & 4 & -1 \\
1 & 3 & 1
\end{bmatrix}.
\]

The characteristic polynomial of the given matrix is

\[
f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 8\lambda^2 - 21\lambda + 18 = (2 - \lambda)(\lambda - 3)^2,
\]

so its eigenvalues are \(\lambda = 2, 3, 3\). The corresponding null spaces are easily found to be

\[
\mathcal{N}(A - 2I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 3I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.
\]

This implies that \(A\) is not diagonalisable and that its Jordan form is

\[
J = B^{-1}AB = \begin{bmatrix} 2 & 1 \\
3 & 1 \\
1 & 3 \end{bmatrix}.
\]
To find a Jordan basis, we need to find vectors \( v_1, v_2, v_3 \) such that \( v_1 \) is an eigenvector with eigenvalue \( \lambda = 2 \) and \( v_2, v_3 \) is a Jordan chain with eigenvalue \( \lambda = 3 \). In our case, we have

\[
N(A - 3I)^2 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},
\]

so it easily follows that a Jordan basis is provided by the vectors

\[
v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = (A - 3I)v_2 = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}.
\]

3. Suppose \( A \) is a 2 × 2 matrix such that \( A^2 = I_2 \) and let \( J \) be the Jordan form of \( A \). Show that \( J^2 = I_2 \) and use this fact to conclude that \( J \) is diagonal.

Write \( J = B^{-1}AB \) for some invertible matrix \( B \). Since \( A^2 = I_2 \), we must also have

\[
J^2 = B^{-1}AB \cdot B^{-1}AB = B^{-1}A^2B = B^{-1}I_2B = I_2.
\]

Next, we show that \( J \) is diagonal. If that is not the case, then \( J \) is a 2 × 2 block and

\[
J = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} \implies J^2 = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda \\ 2\lambda & \lambda^2 \end{bmatrix}.
\]

Comparing the last two equations now gives \( \lambda^2 = 1 \) and \( 2\lambda = 0 \), a contradiction.

4. Suppose \( A \) is a 4 × 4 matrix with characteristic polynomial \( f(\lambda) = \lambda^3(\lambda - 1) \) and suppose its column space is two-dimensional. Find the Jordan form of \( A \).

When it comes to the triple eigenvalue \( \lambda = 0 \), the number of Jordan blocks is

\[
\dim N(A - \lambda I) = \dim N(A) = 4 - \dim C(A) = 2.
\]

In particular, there is one 2 × 2 block and one 1 × 1 block, so the Jordan form is

\[
J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ & & 0 \\ & & & 1 \end{bmatrix}.
\]