**Bilinear forms**

**Definition 3.1 – Bilinear form**

A bilinear form on a real vector space $V$ is a function $f : V \times V \to \mathbb{R}$ which assigns a number to each pair of elements of $V$ in such a way that $f$ is linear in each variable.

- A typical example of a bilinear form is the dot product on $\mathbb{R}^n$.
- We shall usually write $\langle x, y \rangle$ instead of $f(x, y)$ for simplicity and we shall also identify each $1 \times 1$ matrix with its unique entry.

**Theorem 3.2 – Bilinear forms on $\mathbb{R}^n$**

Every bilinear form on $\mathbb{R}^n$ has the form

$$
\langle x, y \rangle = x^t A y = \sum_{i,j} a_{ij} x_i y_j
$$

for some $n \times n$ matrix $A$ and we also have $a_{ij} = \langle e_i, e_j \rangle$ for all $i, j$. 

<table>
<thead>
<tr>
<th>Definition 3.1 – Bilinear form</th>
</tr>
</thead>
<tbody>
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$$
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$$

for some $n \times n$ matrix $A$ and we also have $a_{ij} = \langle e_i, e_j \rangle$ for all $i, j$. |
### Definition 3.3 – Matrix of a bilinear form

Suppose that $\langle \cdot, \cdot \rangle$ is a bilinear form on $V$ and let $v_1, v_2, \ldots, v_n$ be a basis of $V$. The matrix of the form with respect to this basis is the matrix $A$ whose entries are given by $a_{ij} = \langle v_i, v_j \rangle$ for all $i, j$.

### Theorem 3.4 – Change of basis

Suppose that $\langle \cdot, \cdot \rangle$ is a bilinear form on $\mathbb{R}^n$ and let $A$ be its matrix with respect to the standard basis. Then the matrix of the form with respect to some other basis $v_1, v_2, \ldots, v_n$ is given by $B^t A B$, where $B$ is the matrix whose columns are the vectors $v_1, v_2, \ldots, v_n$.

- There is a similar result for linear transformations: if $A$ is the matrix with respect to the standard basis and $v_1, v_2, \ldots, v_n$ is some other basis, then the matrix with respect to the other basis is $B^{-1} A B$. 
Let $P_2$ denote the space of real polynomials of degree at most 2. Then $P_2$ is a vector space and its standard basis is $1, x, x^2$. We can define a bilinear form on $P_2$ by setting

$$\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$$

for all $f, g \in P_2$.

By definition, the matrix of a form with respect to a given basis has entries $a_{ij} = \langle v_i, v_j \rangle$. In our case, $v_i = x^{i-1}$ for each $i$ and so

$$a_{ij} = \langle x^{i-1}, x^{j-1} \rangle = \int_0^1 x^{i+j-2} \, dx = \frac{1}{i + j - 1}.$$

Thus, the matrix of the form with respect to the standard basis is

$$A = \begin{bmatrix}
1 & 1/2 & 1/3 \\
1/2 & 1/3 & 1/4 \\
1/3 & 1/4 & 1/5
\end{bmatrix}.$$
A bilinear form $\langle , \rangle$ on a real vector space $V$ is positive definite, if

$$\langle \mathbf{v}, \mathbf{v} \rangle > 0 \quad \text{for all } \mathbf{v} \neq 0.$$ 

A real $n \times n$ matrix $A$ is positive definite, if $\mathbf{x}^t A \mathbf{x} > 0$ for all $\mathbf{x} \neq 0$.

- A bilinear form on $V$ is positive definite if and only if the matrix of the form with respect to some basis of $V$ is positive definite.
- A positive definite form on $\mathbb{R}^n$ is given by the dot product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i \quad \implies \quad \langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{n} x_i^2.$$ 

- A positive definite form on $P_n$ is given by the formula

$$\langle f, g \rangle = \int_{a}^{b} f(x)g(x) \, dx \quad \implies \quad \langle f, f \rangle = \int_{a}^{b} f(x)^2 \, dx.$$
Consider the bilinear form on $\mathbb{R}^2$ which is defined by

$$\langle x, y \rangle = x_1 y_1 - 2x_1 y_2 - 2x_2 y_1 + 5x_2 y_2.$$  

To check if it is positive definite, we complete the square to get

$$\langle x, x \rangle = x_1^2 - 4x_1 x_2 + 5x_2^2 = (x_1 - 2x_2)^2 + x_2^2.$$  

It now easily follows that the given form is positive definite.

Consider the bilinear form on $\mathbb{R}^2$ which is defined by

$$\langle x, y \rangle = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 3x_2 y_2.$$  

Completing the square as before, one finds that

$$\langle x, x \rangle = x_1^2 + 4x_1 x_2 + 3x_2^2 = (x_1 + 2x_2)^2 - x_2^2.$$  

In particular, $\langle x, x \rangle$ is negative whenever $x_1 = -2x_2$ and $x_2 \neq 0$.  


Symmetric forms

**Definition 3.6 – Symmetric**

A bilinear form \( \langle \cdot, \cdot \rangle \) on a real vector space \( V \) is called symmetric, if

\[
\langle v, w \rangle = \langle w, v \rangle \quad \text{for all } v, w \in V.
\]

A real square matrix \( A \) is called symmetric, if \( a_{ij} = a_{ji} \) for all \( i, j \).

- A bilinear form on \( V \) is symmetric if and only if the matrix of the form with respect to some basis of \( V \) is symmetric.
- A real square matrix \( A \) is symmetric if and only if \( A^t = A \).

**Definition 3.7 – Inner product**

An inner product on a real vector space \( V \) is a bilinear form which is both positive definite and symmetric.
Suppose that \( \langle , \rangle \) is an inner product on a real vector space \( V \).

Then one may define the length of a vector \( v \in V \) by setting

\[
\|v\| = \sqrt{\langle v, v \rangle}
\]

and the angle \( \theta \) between two vectors \( v, w \in V \) by setting

\[
\cos \theta = \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}.
\]

These formulas are known to hold for the inner product on \( \mathbb{R}^n \).

**Theorem 3.8 – Cauchy-Schwarz inequality**

When \( V \) is a real vector space with an inner product, one has

\[
|\langle v, w \rangle| \leq \|v\| \cdot \|w\| \quad \text{for all } v, w \in V.
\]
Orthogonal vectors

**Definition 3.9 – Orthogonal and orthonormal**

Suppose $\langle \cdot , \cdot \rangle$ is a symmetric bilinear form on a real vector space $V$. Two vectors $u, v$ are called orthogonal, if $\langle u, v \rangle = 0$. A basis $v_1, v_2, \ldots, v_n$ of $V$ is called orthogonal, if $\langle v_i, v_j \rangle = 0$ whenever $i \neq j$ and it is called orthonormal, if it is orthogonal with $\langle v_i, v_i \rangle = 1$ for all $i$.

**Theorem 3.10 – Linear combinations**

Let $v_1, v_2, \ldots, v_n$ be an orthogonal basis of an inner product space $V$. Then every vector $v \in V$ can be expressed as a linear combination

$$v = \sum_{i=1}^{n} c_i v_i,$$

where $c_i = \frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle}$ for all $i$.

If the basis is actually orthonormal, then $c_i = \langle v, v_i \rangle$ for all $i$. 
Suppose that \(v_1, v_2, \ldots, v_n\) is a basis of an inner product space \(V\). Then we can find an orthogonal basis \(w_1, w_2, \ldots, w_n\) as follows.

Define the first vector by \(w_1 = v_1\) and the second vector by

\[
\begin{align*}
w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1.
\end{align*}
\]

Then \(w_1, w_2\) are orthogonal and have the same span as \(v_1, v_2\).

Proceeding by induction, suppose \(w_1, w_2, \ldots, w_k\) are orthogonal and have the same span as \(v_1, v_2, \ldots, v_k\). Once we then define

\[
\begin{align*}
w_{k+1} &= v_{k+1} - \sum_{i=1}^{k} \frac{\langle v_{k+1}, w_i \rangle}{\langle w_i, w_i \rangle} w_i,
\end{align*}
\]

we end up with vectors \(w_1, w_2, \ldots, w_{k+1}\) which are orthogonal and have the same span as the original vectors \(v_1, v_2, \ldots, v_{k+1}\).

Using the formula from the last step repeatedly, one may thus obtain an orthogonal basis \(w_1, w_2, \ldots, w_n\) for the vector space \(V\).
We find an orthogonal basis of \( \mathbb{R}^3 \), starting with the basis

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, & \mathbf{v}_2 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, & \mathbf{v}_3 &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.
\end{align*}
\]

We define the first vector by \( \mathbf{w}_1 = \mathbf{v}_1 \) and the second vector by

\[
\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

Then \( \mathbf{w}_1, \mathbf{w}_2 \) are orthogonal and we may define the third vector by

\[
\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \mathbf{w}_2
\]

\[
= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.
\]
Bilinear forms over a complex vector space

Bilinear forms are defined on a complex vector space in the same way that they are defined on a real vector space. However, one needs to conjugate one of the variables to ensure positivity of the dot product.

The complex transpose of a matrix is denoted by $A^* = \overline{A^t}$ and it is also known as the adjoint of $A$. One has $x^*x \geq 0$ for all $x \in \mathbb{C}^n$.

<table>
<thead>
<tr>
<th>Bilinear forms on $\mathbb{R}^n$</th>
<th>Bilinear forms on $\mathbb{C}^n$</th>
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<tbody>
<tr>
<td>Linear in the first variable</td>
<td>Conjugate linear in the first variable</td>
</tr>
<tr>
<td>$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$</td>
<td>$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$</td>
</tr>
<tr>
<td>$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$</td>
<td>$\langle \lambda u, v \rangle = \overline{\lambda} \langle u, v \rangle$</td>
</tr>
<tr>
<td>Linear in the second variable</td>
<td>Linear in the second variable</td>
</tr>
<tr>
<td>$\langle x, y \rangle = x^t A y$ for some $A$</td>
<td>$\langle x, y \rangle = x^* A y$ for some $A$</td>
</tr>
<tr>
<td>Symmetric, if $A^t = A$</td>
<td>Hermitian, if $A^* = A$</td>
</tr>
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<td>Symmetric, if $a_{ij} = a_{ji}$</td>
<td>Hermitian, if $a_{ij} = \overline{a_{ji}}$</td>
</tr>
</tbody>
</table>
**Theorem 3.11 – Inner product and matrices**

Letting $\langle x, y \rangle = x^* y$ be the standard inner product on $\mathbb{C}^n$, one has

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \text{and} \quad \langle x, Ay \rangle = \langle A^* x, y \rangle$$

for any $n \times n$ complex matrix $A$. In fact, these formulas also hold for the standard inner product on $\mathbb{R}^n$, in which case $A^*$ reduces to $A^t$.

**Theorem 3.12 – Eigenvalues of a real symmetric matrix**

The eigenvalues of a real symmetric matrix are all real.

**Theorem 3.13 – Eigenvectors of a real symmetric matrix**

The eigenvectors of a real symmetric matrix corresponding to distinct eigenvalues are necessarily orthogonal to one another.
Orthogonal matrices

**Definition 3.14 – Orthogonal matrix**

A real $n \times n$ matrix $A$ is called orthogonal, if $A^t A = I_n$.

**Theorem 3.15 – Properties of orthogonal matrices**

1. To say that an $n \times n$ matrix $A$ is orthogonal is to say that the columns of $A$ form an orthonormal basis of $\mathbb{R}^n$.
2. The product of two $n \times n$ orthogonal matrices is orthogonal.
3. Left multiplication by an orthogonal matrix preserves both angles and length. When $A$ is an orthogonal matrix, that is, one has

   $$\langle Ax, Ay \rangle = \langle x, y \rangle$$

   and

   $$||Ax|| = ||x||.$$

• An example of a $2 \times 2$ orthogonal matrix is $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$. 
Theorem 3.16 – Spectral theorem

Every real symmetric matrix $A$ is diagonalisable. In fact, there exists an orthogonal matrix $B$ such that $B^{-1}AB = B^tAB$ is diagonal.

1. When the eigenvalues of $A$ are distinct, the eigenvectors of $A$ are orthogonal and we may simply divide each of them by its length to obtain an orthonormal basis of $\mathbb{R}^n$. Such a basis can be merged to form an orthogonal matrix $B$ such that $B^{-1}AB$ is diagonal.

2. When the eigenvalues of $A$ are not distinct, the eigenvectors of $A$ may not be orthogonal. In that case, one may use the Gram-Schmidt procedure to replace eigenvectors that have the same eigenvalue with orthogonal eigenvectors that have the same eigenvalue.

3. The converse of the spectral theorem is also true. That is, if $B$ is an orthogonal matrix and $B^tAB$ is diagonal, then $A$ is symmetric.
Consider the real symmetric matrix
\[ A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}. \]

Its eigenvalues $\lambda = 0, 4, -2$ are distinct and its eigenvectors are
\[ v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \]

Since $v_1, v_2, v_3$ are orthogonal, dividing each of them by its length gives an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors. Then
\[ B = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \]
is an orthogonal matrix such that $B^{-1}AB = B^tAB$ is diagonal.
Orthogonal diagonalisation: Example 2

- Consider the real symmetric matrix
  \[
  A = \begin{bmatrix}
  2 & 1 & 1 \\
  1 & 2 & 1 \\
  1 & 1 & 2
  \end{bmatrix}.
  \]

- Its eigenvalues are \(\lambda = 1, 1, 4\) and its eigenvectors are
  \[
  v_1 = \begin{bmatrix}
  -1 \\
  0 \\
  1
  \end{bmatrix}, \quad v_2 = \begin{bmatrix}
  -1 \\
  1 \\
  0
  \end{bmatrix}, \quad v_3 = \begin{bmatrix}
  1 \\
  1
  \end{bmatrix}.
  \]

- In this case, we use the Gram-Schmidt procedure to replace \(v_1, v_2\) by two orthogonal eigenvectors \(w_1, w_2\). Dividing each of \(w_1, w_2, v_3\) by its length, we then obtain the columns of the orthogonal matrix
  \[
  B = \begin{bmatrix}
  -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\
  0 & 2/\sqrt{6} & 1/\sqrt{3} \\
  1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}
  \end{bmatrix}.
  \]
Definition 3.17 – Quadratic form

A quadratic form in $n$ variables is a function that has the form

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i \leq j} a_{ij}x_ix_j.$$ 

This can be written as $Q(x) = x^tAx$ for some symmetric matrix $A$.

- Here, one needs to be careful with the off-diagonal entries $a_{ij}$, as the coefficient of $x_ix_j$ needs to be halved whenever $i \neq j$. For instance,

$$Q(x) = x_1^2 + 4x_1x_2 + 3x_2^2 = x^tAx, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$ 

- The most general quadratic function in $n$ variables has the form

$$Q(x) = \sum_{i \leq j} a_{ij}x_ix_j + \sum_k b_kx_k + c = x^tAx + b^tx + c.$$
Theorem 3.18 – Diagonalisation of quadratic forms

Let \( Q(x) = x^t A x \) for some symmetric \( n \times n \) matrix \( A \). Then there exists an orthogonal change of variables \( x = B y \) such that

\[
Q(x) = \sum_{i \leq j} a_{ij} x_i x_j = \sum_{i=1}^{n} \lambda_i y_i^2,
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the eigenvalues of the matrix \( A \).

Definition 3.19 – Signature of a quadratic form

The signature of a quadratic form \( Q(x) = x^t A x \) is defined as the pair of integers \( (n_+, n_-) \), where \( n_+ \) is the number of positive eigenvalues of \( A \) and \( n_- \) is the number of negative eigenvalues of \( A \).
We diagonalise the quadratic form

\[ Q(x) = 5x_1^2 + 4x_1x_2 + 2x_2^2 = x^t A x, \quad A = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}. \]

The eigenvalues \( \lambda = 1, 6 \) are distinct and one can easily check that

\[ B = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \implies B^t A B = \begin{bmatrix} 1 & 6 \end{bmatrix}. \]

As usual, the columns of \( B \) were obtained by finding the eigenvectors of \( A \) and by dividing each eigenvector by its length.

Changing variables by \( x = B y \), we now get \( y = B^t x \) and also

\[ y_1^2 + 6y_2^2 = \left( \frac{x_1 - 2x_2}{\sqrt{5}} \right)^2 + 6 \left( \frac{2x_1 + x_2}{\sqrt{5}} \right)^2 = Q(x). \]

This is the change of variables which is asserted by Theorem 3.18.
The following conditions are equivalent for a symmetric matrix $A$.

1. One has $x^tAx > 0$ for all $x \neq 0$.
2. The eigenvalues of $A$ are all positive.
3. One has $\det A_k > 0$ for all $k \times k$ upper left submatrices $A_k$.

The last condition is known as Sylvester’s criterion. When it comes to a $3 \times 3$ matrix, for instance, it refers to the three submatrices

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}.$$

We say that $A$ is negative definite, if $x^tAx < 0$ for all $x \neq 0$. Thus, a negative definite symmetric matrix $A$ has negative eigenvalues and its upper left submatrices $A_k$ are such that $(-1)^k \det A_k > 0$ for all $k$. 

Sylvester’s criterion: Example

Let \( a \) be a real parameter and consider the matrix

\[
A = \begin{bmatrix}
  a & 1 & 1 \\
  1 & 1 & a \\
  1 & a & 5
\end{bmatrix}.
\]

By Sylvester’s criterion, \( A \) is positive definite if and only if

\[
a > 0, \quad \det \begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix} > 0, \quad \det A > 0.
\]

The first two conditions give \( a > 0 \) and \( a > 1 \), while

\[
\det A = -a^3 + 7a - 6 = -(a - 1)(a - 2)(a + 3).
\]

It easily follows that \( A \) is positive definite if and only if \( 1 < a < 2 \).
Given a function $f(x, y)$ of two variables, its directional derivative in the direction of a unit vector $u$ is given by $D_u f = u_1 f_x + u_2 f_y$.

In particular, the second derivative of $f$ in the direction of $u$ is

$$D_u D_u f = u_1 (u_1 f_x + u_2 f_y) x + u_2 (u_1 f_x + u_2 f_y) y$$

$$= u_1^2 f_{xx} + u_1 u_2 f_{yx} + u_2 u_1 f_{xy} + u_2^2 f_{yy} = u^t A u.$$

This computation allows us to classify the critical points of $f$. If the second derivative is positive for all $u \neq 0$, then the function is convex in all directions and we get a local minimum. If the second derivative is negative for all $u \neq 0$, then we get a local maximum.

To classify the critical points, one looks at the Hessian matrix

$$A = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}.$$

This is symmetric, so it is diagonalisable with real eigenvalues. Once we now consider three cases, we obtain the second derivative test.
Application 2: Min/Max value on the unit sphere

Let $A$ be a symmetric $n \times n$ matrix and consider the quadratic form

$$Q(x) = \sum_{i \leq j} a_{ij} x_i x_j = x^t A x.$$  

To find the minimum value of $Q(x)$ on the unit sphere $||x|| = 1$, we let $B$ be an orthogonal matrix such that $B^t A B$ is diagonal and then use the orthogonal change of variables $x = B y$ to write

$$Q(x) = \sum_{i=1}^n \lambda_i y_i^2.$$  

Since $||y|| = ||B y|| = ||x|| = 1$ by orthogonality, we find that

$$Q(x) = \sum_{i=1}^n \lambda_i y_i^2 \geq \sum_{i=1}^n \lambda_{\min} y_i^2 = \lambda_{\min}.$$  

In particular, $\min Q(x)$ is the smallest eigenvalue of $A$, while a similar argument shows that $\max Q(x)$ is the largest eigenvalue of $A$. 


Application 3: Min/Max value of quadratics

- Every quadratic function of \( n \) variables can be expressed in the form

\[
Q(x) = \sum_{i \leq j} a_{ij}x_i x_j + \sum b_k x_k + c = x^t A x + x^t b + c.
\]

- Suppose \( A \) is positive definite symmetric and let \( x_0 = -\frac{1}{2} A^{-1} b \). Then

\[
Q(x_0) = x_0^t A x_0 + x_0^t b + c = -x_0^t A x_0 + c
\]

is the minimum value that is attained by the quadratic because

\[
0 \leq (x - x_0)^t A (x - x_0) = x^t A x - 2x^t A x_0 + x_0^t A x_0
\]

\[
= x^t A x + x^t b + c - Q(x_0)
\]

\[
= Q(x) - Q(x_0).
\]

- When \( A \) is negative definite symmetric, the inequality is reversed and thus \( Q(x_0) \) is the maximum value that is attained by the quadratic.