Chapter 2. Jordan forms

Lecture notes for MA1212

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Generalised eigenvectors

Definition 2.1 – Generalised eigenvector

Suppose λ is an eigenvalue of the square matrix A. We say that $\boldsymbol{v}$ is a generalised eigenvector of A with eigenvalue λ, if $\boldsymbol{v}$ is a nonzero element of the null space of $(A-\lambda I)^j$ for some positive integer j.

- An eigenvector of A with eigenvalue λ is a nonzero element of the null space of $A-\lambda I$, so it is also a generalised eigenvector of A.
- We shall denote by $\mathcal{N}(A-\lambda I)^j$ the null space of $(A-\lambda I)^j$.

Theorem 2.2 – Null spaces eventually stabilise

Let A be a square matrix and let λ be an eigenvalue of A. Then the null spaces $\mathcal{N}(A-\lambda I)^j$ are increasing with j and there is a unique positive integer k such that $\mathcal{N}(A-\lambda I)^j = \mathcal{N}(A-\lambda I)^k$ for all $j \geq k$.

Column space

Definition 2.3 – Column space

The column space of a matrix A is the span of the columns of A. It consists of all vectors $\boldsymbol{y}$ that have the form $\boldsymbol{y} = A\boldsymbol{x}$ for some vector $\boldsymbol{x}$.

- The column space of A is usually denoted by $\mathcal{C}(A)$. The dimension of the column space of A is also known as the rank of A.
- To find a basis for the column space of a matrix A, we first compute its reduced row echelon form R. Then the columns of R that contain pivots form a basis for the column space of R and the corresponding columns of A form a basis for the column space of A.
- The dimension of the column space is the number of pivots and the dimension of the null space is the number of free variables, so

$$\text{rank} + \text{nullity} = \text{number of columns.}$$

Column space: Example

- We find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 5 & 4 \\ 3 & 1 & 7 & 2 & 3 \\ 2 & 1 & 5 & 1 & 5 \end{bmatrix}.$$

- The reduced row echelon form of this matrix is given by

$$R = \begin{bmatrix} \mathbf{1} & 0 & 2 & 0 & 0 \\ 0 & \mathbf{1} & 1 & 0 & 7 \\ 0 & 0 & 0 & \mathbf{1} & -2 \end{bmatrix}.$$

- Since the pivots of R appear in the 1st, 2nd and 4th columns, a basis for the column space of A is formed by the corresponding columns

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_4 = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}.$$

Invariant subspaces

Definition 2.4 – Invariant subspace

Suppose A is an $n \times n$ complex matrix and U is a subspace of $\mathbb{C}^n$. We say that U is A-invariant, if $A\boldsymbol{u} \in U$ for each vector $\boldsymbol{u} \in U$.

- The one-dimensional invariant subspaces correspond to eigenvectors.
- The matrix A might represent a reflection of the xy-plane along a line through the origin. In that case, A has two one-dimensional invariant subspaces. One is spanned by a vector which is parallel to the line of reflection and one is spanned by a vector which is perpendicular to it.
- Similarly, A might represent a rotation in $\mathbb{R}^3$ around some axis. Such a matrix has an one-dimensional invariant subspace along the axis of rotation and a two-dimensional invariant subspace perpendicular to it.
- Both the null spaces $\mathcal{N}(A - \lambda I)^j$ and the column spaces $\mathcal{C}(A - \lambda I)^j$ give rise to A-invariant subspaces. We shall mostly study the former.

Coordinate vectors

Theorem 2.5 – Coordinate vectors

Suppose B is invertible with columns $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ and A is $n \times n$. Then the kth column of $B^{-1}AB$ lists the coefficients that one needs in order to express $A\boldsymbol{v}_k$ as a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$.

- That is, the kth column of $B^{-1}AB$ is the coordinate vector of $A\boldsymbol{v}_k$ with respect to the basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$. This is true because

$$A\boldsymbol{v}_k = \sum_{i=1}^{n} c_i \boldsymbol{v}_i \quad \Longleftrightarrow \quad AB\boldsymbol{e}_k = \sum_{i=1}^{n} c_i B\boldsymbol{e}_i$$
$$\Longleftrightarrow \quad B^{-1}AB\boldsymbol{e}_k = \sum_{i=1}^{n} c_i \boldsymbol{e}_i.$$

- Suppose, for instance, that $A\boldsymbol{v}_k$ is a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Then the kth column of $B^{-1}AB$ is a linear combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, so its first two entries are nonzero and its other entries are zero.

Invariant subspaces and blocks

- Consider a 3×3 matrix A which has two invariant subspaces

$$U = \mathrm{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}, \qquad V = \mathrm{Span}\{\boldsymbol{v}_3\}$$

such that $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis of $\mathbb{R}^3$. We can then write

$$\left\{ \begin{array}{l} A\boldsymbol{v}_1 = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 \\ A\boldsymbol{v}_2 = b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 \\ A\boldsymbol{v}_3 = c\boldsymbol{v}_3 \end{array} \right\}$$

for some scalars a_1, a_2, b_1, b_2, c and this implies the identity

$$B = \left[\boldsymbol{v}_1 \;\; \boldsymbol{v}_2 \mid \boldsymbol{v}_3\right] \quad \Longrightarrow \quad B^{-1}AB = \left[\begin{array}{cc|c} a_1 & b_1 & \\ a_2 & b_2 & \\ \hline & & c \end{array}\right].$$

- Thus, we get a 2×2 block for the 2-dimensional invariant subspace and an 1×1 block for the 1-dimensional invariant subspace.

Jordan chains

Definition 2.6 – Jordan chain

Suppose λ is an eigenvalue of the square matrix A. We say that the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ form a Jordan chain, if they are nonzero with

$$(A - \lambda I)\boldsymbol{v}_i = \left\{ \begin{array}{cl} \boldsymbol{v}_{i+1} & \text{when } i < k \\ 0 & \text{when } i = k \end{array} \right\}.$$

- The last vector in a Jordan chain is simply an eigenvector of A.
- The first vector in a Jordan chain of length k is a vector that lies in the null space $\mathcal{N}(A - \lambda I)^k$ but not in the null space $\mathcal{N}(A - \lambda I)^{k-1}$.
- The span of a Jordan chain is an A-invariant subspace because

  $$A\boldsymbol{v}_i = (\lambda I + A - \lambda I)\boldsymbol{v}_i = \lambda \boldsymbol{v}_i + \boldsymbol{v}_{i+1}$$

  when $i < k$ and since $A\boldsymbol{v}_k = \lambda \boldsymbol{v}_k$. In particular, each $A\boldsymbol{v}_i$ except for the last one is a linear combination of precisely two vectors.

Jordan chains: Example 1

N_1	•	•	•	$\dim N_1 = 3$
N_2	•	•	•	$\dim N_2 - \dim N_1 = 3$
N_3	•			$\dim N_3 - \dim N_2 = 1$

- Let λ be an eigenvalue of A and let $N_j = \mathcal{N}(A - \lambda I)^j$ for each j. As we already know, these null spaces are increasing with j. Assume, for instance, that $\dim N_1 = 3$, $\dim N_2 = 6$ and $\dim N_3 = 7$.
- We draw a diagram by placing 3 dots in the first row, $6 - 3 = 3$ dots in the second row and $7 - 6 = 1$ dot in the third row.
- The dots in this diagram represent linearly independent vectors and if a dot represents $\boldsymbol{v}$, then the dot right above it represents $(A - \lambda I)\boldsymbol{v}$.
- Reading the diagram vertically, we conclude that there is one Jordan chain of length 3 as well as two Jordan chains of length 2.

Jordan chains: Example 2

N_1	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\dim N_1 = 4$
N_2	$\bullet$	$\bullet$			$\dim N_2 - \dim N_1 = 2$
N_3	$\bullet$				$\dim N_3 - \dim N_2 = 1$

- Let λ be an eigenvalue of A and let $N_j = \mathcal{N}(A - \lambda I)^j$ for each j. In this example, we assume $\dim N_1 = 4$, $\dim N_2 = 6$ and $\dim N_3 = 7$.
- The corresponding diagram includes 4 dots in the first row, $6 - 4 = 2$ dots in the second row and $7 - 6 = 1$ dot in the third row.
- Reading the diagram vertically, we get one Jordan chain of length 3, one Jordan chain of length 2 and two Jordan chains of length 1.
- Jordan chains of length 1 are just eigenvectors of A. To find a Jordan chain of length 3, one needs to find a vector $\boldsymbol{v}$ that lies in N_3 but not in N_2. Such a vector generates the chain $\boldsymbol{v}, (A - \lambda I)\boldsymbol{v}, (A - \lambda I)^2\boldsymbol{v}$.

Jordan blocks and Jordan form

Definition 2.7 – Jordan blocks and Jordan form

A Jordan block with eigenvalue λ is a square matrix whose entries are equal to λ on the diagonal, equal to 1 right below the diagonal and equal to 0 elsewhere. A Jordan form is a block diagonal matrix that consists entirely of Jordan blocks.

- Some typical examples of Jordan blocks are

$$J_1 = \begin{bmatrix} \lambda \end{bmatrix}, \qquad J_2 = \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix}, \qquad J_3 = \begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix}.$$

- Two typical examples of Jordan forms are

$$J = \left[\begin{array}{ccc|c} 2 & & & \\ 1 & 2 & & \\ & 1 & 2 & \\ \hline & & & 2 \end{array}\right], \qquad J' = \left[\begin{array}{cc|c|c} 1 & & & \\ 1 & 1 & & \\ \hline & & 1 & \\ \hline & & & 2 \end{array}\right].$$

Jordan chains $\longleftrightarrow$ Jordan blocks

Theorem 2.8 – Jordan chains and Jordan blocks

Suppose A is an $n \times n$ complex matrix and let B be a matrix whose columns form a basis of $\mathbb{C}^n$ consisting entirely of Jordan chains of A. Then $J = B^{-1}AB$ is a matrix in Jordan form whose kth Jordan block has the same size and the same eigenvalue as the kth Jordan chain.

- For instance, suppose A is 4×4 with eigenvalues $\lambda = 0, 0, 3, 3$. If A has a Jordan chain of length 2 with $\lambda = 0$ and two Jordan chains of length 1 with $\lambda = 3$, then the Jordan form of A is

$$J = \left[\begin{array}{cc|c|c} 0 & & & \\ 1 & 0 & & \\ \hline & & 3 & \\ \hline & & & 3 \end{array}\right].$$

- The Jordan form of a square matrix is unique (up to a permutation of its blocks). There might be several blocks with the same eigenvalue.

Jordan form: Example 1

- We compute the Jordan form of the matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}.$$

- The characteristic polynomial is $f(\lambda) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, so the only eigenvalue is $\lambda = 2$. Moreover, it is easy to check that

$$\dim \mathcal{N}(A - 2I) = 1, \qquad \dim \mathcal{N}(A - 2I)^2 = 2.$$

- The corresponding diagram for the Jordan chains is $\begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}$ and we need to find a Jordan chain of length 2. Pick a vector $\boldsymbol{v}_1$ that lies in the null space of $(A - 2I)^2$ but not in the null space of $A - 2I$.
- Letting $\boldsymbol{v}_2 = (A - 2I)\boldsymbol{v}_1$ now gives a Jordan chain $\boldsymbol{v}_1, \boldsymbol{v}_2$ and so

$$B = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} \quad \Longrightarrow \quad J = B^{-1}AB = \begin{bmatrix} 2 & \\ 1 & 2 \end{bmatrix}.$$

Jordan form: Example 2, page 1

- We compute the Jordan form of the matrix

$$A = \begin{bmatrix} 4 & -5 & 2 \\ 1 & -2 & 2 \\ 2 & -6 & 5 \end{bmatrix}.$$

- In this case, the characteristic polynomial is given by

$$f(\lambda) = -\lambda^3 + 7\lambda^2 - 15\lambda + 9 = -(\lambda - 1)(\lambda - 3)^2,$$

  so there are two eigenvalues that need to be treated separately.
- When it comes to the eigenvalue $\lambda = 1$, one finds that

$$\dim \mathcal{N}(A - I) = 1, \qquad \dim \mathcal{N}(A - I)^2 = 1.$$

- The corresponding diagram for the Jordan chains is $\bullet$ and it only contains one Jordan chain of length 1. In fact, a Jordan chain of length 1 is merely an eigenvector $\boldsymbol{v}_1$ with eigenvalue $\lambda = 1$.

Jordan form: Example 2, page 2

- When it comes to the eigenvalue $\lambda = 3$, one finds that

$$\dim \mathcal{N}(A - 3I) = 1, \qquad \dim \mathcal{N}(A - 3I)^2 = 2.$$

- The corresponding diagram for the Jordan chains is $\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}$ and we need to find a Jordan chain of length 2. Pick a vector $\boldsymbol{v}_2$ that lies in the null space of $(A - 3I)^2$ but not in the null space of $A - 3I$. Such a vector gives rise to a Jordan chain $\boldsymbol{v}_2, \boldsymbol{v}_3$ with $\boldsymbol{v}_3 = (A - 3I)\boldsymbol{v}_2$.
- In particular, A has a Jordan chain $\boldsymbol{v}_1$ with eigenvalue $\lambda = 1$ and also a Jordan chain $\boldsymbol{v}_2, \boldsymbol{v}_3$ with eigenvalue $\lambda = 3$, so its Jordan form is

$$B = \big[\boldsymbol{v}_1 \mid \boldsymbol{v}_2 \;\; \boldsymbol{v}_3\big] \quad \Longrightarrow \quad J = B^{-1}AB = \left[\begin{array}{c|cc} 1 & & \\ \hline & 3 & \\ & 1 & 3 \end{array}\right].$$

- The chosen vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ are by no means unique. In fact, there are infinitely many matrices B such that $B^{-1}AB$ is in Jordan form.

Jordan basis and Jordan form

Definition 2.9 – Jordan basis

A Jordan basis for an $n \times n$ complex matrix A is a basis of $\mathbb{C}^n$ that consists entirely of Jordan chains of A.

1. **Finding the Jordan form**. Determine the various eigenvalues λ and apply the following steps for each λ. Compute the numbers

$$d_j = \dim \mathcal{N}(A - \lambda I)^j$$

until they stabilise and draw a diagram for the Jordan chains. The lengths of these chains represent the sizes of the Jordan blocks.

2. **Finding a Jordan basis**. Consult the diagram of Jordan chains for each eigenvalue λ and worry about the longest chains first. To find a chain of length $k > 1$, pick a vector that lies in the kth null space but not in the previous one and repeatedly multiply it by $A - \lambda I$. Once you have a Jordan chain, you may proceed similarly to find the next longest chain. Chains of length 1 are merely eigenvectors of A.

Number of Jordan chains

Theorem 2.10 – Number of Jordan chains

Consider the diagram of Jordan chains for an eigenvalue λ which has multiplicity m as a root of the characteristic polynomial of A.

1. The total number of dots in the diagram is equal to m.
2. The total number of Jordan chains is equal to $\dim \mathcal{N}(A - \lambda I)$.

- The first number is also known as the algebraic multiplicity of λ. The second number is also known as the geometric multiplicity of λ.
- When $m \leq 3$, one only needs to know these two numbers to find the diagram of Jordan chains (and thus the sizes of the Jordan blocks).
- For instance, the diagram for a simple eigenvalue λ contains only one dot, so each simple eigenvalue λ contributes a single 1×1 block.
- Similarly, a triple eigenvalue λ such that $\dim \mathcal{N}(A - \lambda I) = 2$ has a diagram containing 3 dots but only two chains. Such an eigenvalue will necessarily contribute one 2×2 block and one 1×1 block.

Direct sums

Theorem 2.11 – Linear independence of Jordan chains

Suppose $\gamma_1, \gamma_2, \ldots, \gamma_m$ are Jordan chains of a square matrix A. If the last vectors of the Jordan chains are linearly independent, then all the vectors that belong to the Jordan chains are linearly independent.

Definition 2.12 – Direct sum

Let U, V be subspaces of a vector space W. Their sum $U + V$ is the set of all vectors $\boldsymbol{w}$ which have the form $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ for some $\boldsymbol{u} \in U$ and some $\boldsymbol{v} \in V$. If it happens that $U \cap V = \{0\}$, then we say that the sum is direct and we denote it by $U \oplus V$.

Theorem 2.13 – Basis of a direct sum

One may obtain a basis for the direct sum $U \oplus V$ by appending a basis of V to a basis of U. In particular, $\dim(U \oplus V) = \dim U + \dim V$.

Primary decomposition theorem

Theorem 2.14 – Primary decomposition theorem

Given an $n \times n$ complex matrix A, one can write

$$\mathbb{C}^n = \mathcal{N}(A - \lambda_1 I)^{k_1} \oplus \cdots \oplus \mathcal{N}(A - \lambda_p I)^{k_p},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the distinct eigenvalues of A and each k_i is the exponent at which the null spaces $\mathcal{N}(A - \lambda_i I)^j$ stabilise.

- This theorem is mostly of theoretical value. It ensures that one can always find a basis of $\mathbb{C}^n$ by looking at the various null spaces.
- Each of the null spaces has a basis which consists entirely of Jordan chains. Since the above sum is direct, we may then merge all these Jordan chains to obtain a Jordan basis for the given matrix A.
- Letting B denote the matrix whose columns form a Jordan basis, we conclude that B is invertible, while $J = B^{-1}AB$ is in Jordan form.

Similar matrices

Definition 2.15 – Similar matrices

A square matrix A is said to be similar to a square matrix C, if there exists an invertible matrix B such that $C = B^{-1}AB$.

Theorem 2.16 – Similarities of similar matrices

If two square matrices A and C are similar, then

1. They have the same characteristic polynomial and eigenvalues.
2. They have the same rank, nullity, trace and determinant.
3. The matrices A^n and C^n are similar for any positive integer n.

Theorem 2.17 – Similarity test

Two square matrices are similar if and only if their Jordan forms are the same (up to a permutation of their Jordan blocks).

Properties of Jordan blocks

Theorem 2.18 – Properties of Jordan blocks

Suppose that J is a $k \times k$ Jordan block with eigenvalue λ.

1. The entries of $(J - \lambda I)^j$ are equal to 1, if they lie j steps below the diagonal, and they are equal to 0, otherwise.
2. One has $(J - \lambda I)^j = 0$ if and only if $j \geq k$.

- Loosely speaking, the powers of $J - \lambda I$ are obtained by shifting its entries downwards one step at a time. For instance, one has

$$J - \lambda I = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{bmatrix} \quad \Longrightarrow \quad (J - \lambda I)^2 = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 1 & 0 & 0 & \\ & 1 & 0 & 0 \end{bmatrix}.$$

- When it comes to a Jordan block J, it is easy to see that the null spaces $\mathcal{N}(J - \lambda I)^j$ are increasing until they eventually stabilise.

Powers of Jordan blocks

Theorem 2.19 – Powers of Jordan blocks

Let J be a Jordan block with eigenvalue $\lambda \neq 0$. Then the entries of its nth power J^n are equal to λ^n on the diagonal, $\binom{n}{1}\lambda^{n-1}$ right below the diagonal, $\binom{n}{2}\lambda^{n-2}$ two steps below the diagonal, and so on.

- For instance, if J is a 2×2 Jordan block with eigenvalue $\lambda \neq 0$, then

$$J = \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix} \quad \Longrightarrow \quad J^n = \begin{bmatrix} \lambda^n & \\ n\lambda^{n-1} & \lambda^n \end{bmatrix}.$$

- Jordan blocks with eigenvalue $\lambda = 0$ are somewhat different, as

$$J = \begin{bmatrix} 0 & & \\ 1 & 0 & \\ & 1 & 0 \end{bmatrix} \quad \Longrightarrow \quad J^2 = \begin{bmatrix} 0 & & \\ 0 & 0 & \\ 1 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad J^3 = 0.$$

In particular, powers of such matrices may be computed as before by shifting their entries downwards one step at a time.

Powers of a square matrix

1. Given a complex square matrix A, we first find its Jordan form J as well as a matrix B such that $J = B^{-1}AB$ is in Jordan form.
2. Letting $J_1, J_2, \ldots, J_k$ denote the Jordan blocks of J, one has

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{bmatrix} \implies J^n = \begin{bmatrix} J_1^n & & & \\ & J_2^n & & \\ & & \ddots & \\ & & & J_k^n \end{bmatrix}.$$

   The powers J_i^n are easy to compute, as those are powers of Jordan blocks and we have explicit formulas for computing them.
3. The equation above determines the powers of the Jordan form J. To find the powers of the original matrix A, we note that

$$\begin{aligned} J = B^{-1}AB \quad &\implies \quad J^n = B^{-1}A^nB \\ &\implies \quad A^n = BJ^nB^{-1}. \end{aligned}$$

Matrices and polynomials

Theorem 2.20 – Matrices and polynomials

Let A be a square matrix and let g be a polynomial.

1. The matrix A is a root of g if and only if its Jordan form J is a root of g. In other words, one has $g(A) = 0 \iff g(J) = 0$.
2. If the matrix A is a root of g, then every eigenvalue λ of A is a root of g. In other words, one has $g(A) = 0 \implies g(\lambda) = 0$.

Theorem 2.21 – Cayley-Hamilton theorem

Every square matrix is a root of its characteristic polynomial.

Definition 2.22 – Minimal polynomial

The minimal polynomial $m(\lambda)$ of a square matrix A is defined as the monic polynomial of smallest degree that has A as a root.

Minimal polynomial

Theorem 2.23 – Properties of the minimal polynomial

Let A be a square matrix and let $m(\lambda)$ be its minimal polynomial.

1. The minimal polynomial divides every polynomial that has A as a root. In particular, it divides the characteristic polynomial $f(\lambda)$.
2. If $\lambda_1, \lambda_2, \dots, \lambda_p$ are the distinct eigenvalues of A, then

$$m(\lambda) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_p)^{k_p},$$

   where each exponent k_i is given by the size of the largest Jordan block that corresponds to the eigenvalue λ_i.
3. The matrix A is diagonalisable if and only if the factors of $m(\lambda)$ are all linear, namely if and only if $k_i = 1$ for each i.

- For instance, a square matrix A such that $A^3 = A$ is diagonalisable because its minimal polynomial divides $\lambda^3 - \lambda = \lambda(\lambda - 1)(\lambda + 1)$.

Minimal polynomial: Example

- Suppose A is a matrix with characteristic and minimal polynomials

$$f(\lambda) = -(\lambda - 2)^2(\lambda - 4)^3, \qquad m(\lambda) = (\lambda - 2)(\lambda - 4)^2.$$

- Then A is a 5×5 matrix that has two distinct eigenvalues. Let us find its Jordan form by looking at each eigenvalue separately.
- When it comes to the eigenvalue $\lambda_1 = 2$, we have $k_1 = 1$ and so the largest Jordan block is 1×1. Since $\lambda_1 = 2$ is a double eigenvalue, we must thus have two 1×1 Jordan blocks with this eigenvalue.
- When it comes to the eigenvalue $\lambda_2 = 4$, we have $k_2 = 2$ and so the largest Jordan block is 2×2. Since $\lambda_2 = 4$ is a triple eigenvalue, it contributes one 2×2 Jordan block and one 1×1 Jordan block.
- In other words, the Jordan form of A consists of the blocks

$$\begin{bmatrix} 2 \end{bmatrix}, \qquad \begin{bmatrix} 2 \end{bmatrix}, \qquad \begin{bmatrix} 4 & \\ 1 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 4 \end{bmatrix}.$$