Chapter 1. Diagonalisation

Lecture notes for MA1212

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Recursive relations involving two terms

- A recursive relation expresses each term of a sequence $x_1, x_2, \ldots$ as a function of the previous terms. One generally prefers to have a closed formula, namely one that expresses x_n as a function of n alone.
- For instance, suppose that the sequences x_n, y_n are such that

$$\left\{ \begin{array}{l} x_n = 2x_{n-1} + 4y_{n-1} \\ y_n = 5x_{n-1} + 3y_{n-1} \end{array} \right\}.$$

- Letting $\boldsymbol{u}_n$ denote the vector of unknowns, one can then write

$$\boldsymbol{u}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{u}_n = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = A\boldsymbol{u}_{n-1}$$

  for some 2×2 matrix A. This equation is easily seen to imply that

$$\boldsymbol{u}_n = A\boldsymbol{u}_{n-1} = A^2\boldsymbol{u}_{n-2} = \ldots = A^n\boldsymbol{u}_0.$$

- Thus, one needs to compute A^n in order to obtain a closed formula.

Recursive relations involving three or more terms

- A recursive relation may involve several consecutive terms such as

$$x_{n+3} = x_{n+2} + 5x_{n+1} + 3x_n.$$

  This relation can be handled exactly as before, although one needs to keep track of three terms in order to determine the next one.

- Letting $\boldsymbol{u}_n$ consist of three consecutive terms, we can now write

$$\boldsymbol{u}_n = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ x_{n+3} \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{u}_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ x_{n+1} \\ x_{n+2} \end{bmatrix} = A\boldsymbol{u}_{n-1}$$

  for some 3×3 matrix A. This equation is easily seen to imply that

$$\boldsymbol{u}_n = A\boldsymbol{u}_{n-1} = A^2\boldsymbol{u}_{n-2} = \ldots = A^n\boldsymbol{u}_0.$$

- Once we are able to compute powers of the matrix A, we can then obtain a closed formula for $\boldsymbol{u}_n$ and also one for the sequence x_n.

Powers of diagonalisable matrices

Definition 1.1 – Diagonalisable matrix

A square matrix A is called diagonalisable, if there exists an invertible matrix B such that $D = B^{-1}AB$ is diagonal.

- To compute powers of a diagonal matrix D, one notes that

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_k \end{bmatrix} \quad \Longrightarrow \quad D^n = \begin{bmatrix} \lambda_1^n & & & \\ & \lambda_2^n & & \\ & & \ddots & \\ & & & \lambda_k^n \end{bmatrix}.$$

- To compute powers of a diagonalisable matrix A, one notes that

$$\begin{aligned} D = B^{-1}AB \quad &\Longrightarrow \quad D^n = B^{-1}A^nB \\ &\Longrightarrow \quad A^n = BD^nB^{-1}. \end{aligned}$$

This is an explicit formula that relates powers of A to powers of D.

Eigenvalues and eigenvectors

Theorem 1.2 – Diagonalisation

Let A be an $n \times n$ matrix. Then $B^{-1}AB$ is diagonal with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ if and only if the columns $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ of B are linearly independent vectors such that $A\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$ for each i.

Definition 1.3 – Eigenvalues and eigenvectors

Suppose A is a square matrix. A vector $\boldsymbol{v}$ is called an eigenvector of A with eigenvalue λ, if $\boldsymbol{v}$ is nonzero with $A\boldsymbol{v} = \lambda \boldsymbol{v}$.

- The ith column of a matrix A is given by $A\boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the vector whose ith entry is equal to 1 and all other entries are equal to 0.
- According to the theorem, an $n \times n$ matrix is diagonalisable if and only if it has n linearly independent eigenvectors.
- We shall first focus on finding the eigenvalues λ. Once we know the eigenvalues, we may then find the eigenvectors by solving $A\boldsymbol{v} = \lambda \boldsymbol{v}$.

Finding the eigenvalues of a matrix

Definition 1.4 – Characteristic polynomial

The characteristic polynomial of a square matrix A is defined by

$$f(\lambda) = \det(A - \lambda I).$$

The roots of this polynomial are the eigenvalues of the matrix A.

Theorem 1.5 – Eigenvalues of simple matrices

1. The eigenvalues of a 2×2 matrix A are the roots of

   $$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A,$$

   where $\operatorname{tr} A$ is the trace of A, the sum of its diagonal entries.
2. If A is either a lower triangular or an upper triangular matrix, then the eigenvalues of A are the diagonal entries of A.

Example: A diagonalisable 2×2 matrix, page 1

- We show that the matrix A is diagonalisable in the case that

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}.$$

- We need to check that A has two linearly independent eigenvectors. The eigenvalues of A are the roots of the characteristic polynomial

$$\begin{aligned} f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A &= \lambda^2 - 7\lambda + 6 \\ &= (\lambda - 1)(\lambda - 6). \end{aligned}$$

- When $\lambda = 1$, eigenvectors satisfy the system $A\boldsymbol{v} = \boldsymbol{v}$, so we get

$$(A - I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad x + y = 0.$$

  That is, every eigenvector with eigenvalue $\lambda = 1$ has the form

$$\boldsymbol{v} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad y \neq 0.$$

Example: A diagonalisable 2×2 matrix, page 2

- When $\lambda = 6$, eigenvectors satisfy the system $A\boldsymbol{v} = 6\boldsymbol{v}$, so we get

$$(A - 6I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad 4x - y = 0.$$

  That is, every eigenvector with eigenvalue $\lambda = 6$ has the form

$$\boldsymbol{v} = \begin{bmatrix} x \\ 4x \end{bmatrix} = x \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \qquad x \neq 0.$$

- Since A has two distinct eigenvectors, we may now easily check that

$$B = \begin{bmatrix} -1 & 1 \\ 1 & 4 \end{bmatrix} \quad \Longrightarrow \quad B^{-1}AB = \begin{bmatrix} 1 & \\ & 6 \end{bmatrix}.$$

- This is a typical application of the general theory. If the columns of B are eigenvectors with eigenvalues λ_1, λ_2, then $B^{-1}AB$ is a diagonal matrix whose diagonal entries are λ_1, λ_2.

Example: A non-diagonalisable 2×2 matrix

- We show that the matrix A is not diagonalisable in the case that

$$A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}.$$

- The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2,$$

  so the only eigenvalue is $\lambda = 1$. The eigenvectors satisfy $A\boldsymbol{v} = \boldsymbol{v}$ and

$$(A - I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad x + 2y = 0.$$

- In other words, every eigenvector of A must have the form

$$\boldsymbol{v} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad y \neq 0.$$

  Since we only found one eigenvector, A is not diagonalisable.

Theorem 1.6 – Rational root test

Consider a polynomial $f(x)$ with integer coefficients, say

$$f(x) = a_n x^n + \ldots + a_1 x + a_0.$$

Every rational root of $f(x)$ has the form p/q, where p, q are relatively prime integers such that p divides a_0 and q divides a_n.

Theorem 1.7 – Factor theorem

If a polynomial $f(x)$ has $x = \alpha$ as a root, then it also has $x - \alpha$ as a factor, namely $f(x) = (x - \alpha) \cdot g(x)$ for some polynomial $g(x)$.

- For instance, let $f(x) = x^3 - 3x - 2$. The only possible rational roots are $x = \pm 1$ and $x = \pm 2$. Noting that $x = -1$ is a root, we get

$$f(x) = (x + 1)(x^2 - x - 2) = (x + 1)(x + 1)(x - 2).$$

Example: An integer 3×3 matrix

- As a typical example, we compute the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

- In this case, the characteristic polynomial of A is given by

$$f(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix} = -\lambda^3 + 6\lambda^2 - 9\lambda + 4.$$

- The only possible rational roots are $\lambda = \pm 1, \pm 2, \pm 4$. Going through this list, one easily finds that $\lambda = 1$ is a root and that

$$f(\lambda) = -(\lambda - 1)(\lambda^2 - 5\lambda + 4) = -(\lambda - 1)(\lambda - 1)(\lambda - 4).$$

- Thus, $\lambda = 1$ is a double eigenvalue and $\lambda = 4$ is a simple eigenvalue.

Example: A rational 3×3 matrix

- We apply the same method to find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1/2 & 1/3 & 1 \\ 2 & 1/3 & 1/2 \\ 1 & 1/3 & 1/2 \end{bmatrix}.$$

- In this case, the characteristic polynomial of A is given by

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + \frac{4}{3}\lambda^2 + \frac{5}{4}\lambda + \frac{1}{6}.$$

- Clearing denominators gives $12f(\lambda) = -12\lambda^3 + 16\lambda^2 + 15\lambda + 2$, so the only possible rational roots have the form p/q, where p divides 2 and q divides 12. Noting that $\lambda = 2$ is a root, one finds that

$$\begin{aligned} f(\lambda) &= -(\lambda - 2)(\lambda^2 + 2\lambda/3 + 1/12) \\ &= -(\lambda - 2)(\lambda + 1/6)(\lambda + 1/2). \end{aligned}$$

- Thus, the eigenvalues of A are $\lambda = 2$, $\lambda = -1/6$ and $\lambda = -1/2$.

Null space of a matrix

Definition 1.8 – Null space

The null space of an $m \times n$ matrix A is the set of all vectors $\boldsymbol{v} \in \mathbb{R}^n$ such that $A\boldsymbol{v} = 0$. This is easily seen to be a subspace of $\mathbb{R}^n$.

- The null space of A is usually denoted by $\mathcal{N}(A)$ and its dimension is called the nullity of A. The nullity of A is given by the number of free variables in the reduced row echelon form of A.
- Suppose A is a square matrix and λ is an eigenvalue of A. Then the corresponding eigenvectors $\boldsymbol{v}$ are nonzero vectors such that

$$A\boldsymbol{v} = \lambda\boldsymbol{v} \quad \Longleftrightarrow \quad (A - \lambda I)\boldsymbol{v} = 0.$$

- In particular, eigenvectors of A with eigenvalue λ are merely nonzero elements of $\mathcal{N}(A - \lambda I)$. If the matrix A has several eigenvalues λ, then one needs to determine $\mathcal{N}(A - \lambda I)$ for each λ separately.
- There is a standard method for finding the null space of a matrix.

Finding the null space, page 1

- The null space of a matrix A is equal to the null space of its reduced row echelon form R. Let us determine the latter in the case that
$$R = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -4 & 1 \end{bmatrix}.$$
- To find the null space of R, one needs to solve the system $R\boldsymbol{x} = 0$. Note that the equations of this system can be written as
$$x_1 - 2x_3 + 3x_4 = 0, \qquad x_2 - 4x_3 + x_4 = 0.$$
- Once we now eliminate the leading variables, we conclude that
$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 - 3x_4 \\ 4x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix}.$$
- This equation expresses the vectors $\boldsymbol{x}$ of the null space of R in terms of the two free variables, so the null space of R is two-dimensional.

Finding the null space, page 2

- Next, we find an explicit basis for the null space of R. As we have already seen, every vector in the null space has the form

$$\boldsymbol{x} = \begin{bmatrix} 2x_3 - 3x_4 \\ 4x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} = x_3\boldsymbol{v} + x_4\boldsymbol{w}$$

  for some particular vectors $\boldsymbol{v}, \boldsymbol{w}$. Since the variables x_3, x_4 are both free, this means that the null space of R is the span of $\boldsymbol{v}, \boldsymbol{w}$.
- To check that $\boldsymbol{v}, \boldsymbol{w}$ are linearly independent, suppose $x_3\boldsymbol{v} + x_4\boldsymbol{w} = 0$ for some scalars x_3, x_4. Then we must have $\boldsymbol{x} = 0$ by above. Looking at the last two entries of $\boldsymbol{x}$, we find that $x_3 = x_4 = 0$, as needed.
- To find the null space of an arbitrary matrix A, one may proceed in a similar manner. Let R be the reduced row echelon form, write down the equations for the system $R\boldsymbol{x} = 0$ and then eliminate the leading variables. As above, this will give rise to a basis for $\mathcal{N}(R) = \mathcal{N}(A)$.

Example: A non-diagonalisable 3×3 matrix

- We show that the matrix A is not diagonalisable in the case that

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

- As we have already seen, the characteristic polynomial of A is

$$f(\lambda) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = -(\lambda - 1)^2(\lambda - 4).$$

- In particular, the only eigenvectors of A are nonzero elements of

$$\mathcal{N}(A - I) = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 4I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \right\}.$$

- Since we only found two eigenvectors, A is not diagonalisable.

Example: Complex eigenvalues

- The eigenvalues of a real matrix are not necessarily real and the same is true for the eigenvectors. For instance, consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

- The characteristic polynomial of this matrix is given by

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 2\lambda + 2,$$

so the eigenvalues are $\lambda = 1 \pm i$. As for the eigenvectors, we have

$$A - \lambda I = \begin{bmatrix} \mp i & -1 \\ 1 & \mp i \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \mp i \\ 0 & 0 \end{bmatrix}$$

and the corresponding eigenvectors are nonzero elements of

$$\mathcal{N}(A - \lambda I) = \left\{ \begin{bmatrix} \pm iy \\ y \end{bmatrix} : y \in \mathbb{R} \right\} = \operatorname{Span} \left\{ \begin{bmatrix} \pm i \\ 1 \end{bmatrix} \right\}.$$

- We shall not deal with complex eigenvalues/eigenvectors very much.

Distinct eigenvalues

Theorem 1.9 – Distinct eigenvalues

Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ are eigenvectors of an $n \times n$ matrix A which correspond to distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$. Then $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_k$ are linearly independent. In particular, every $n \times n$ matrix that has n distinct eigenvalues must be diagonalisable.

- For instance, consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Since the matrix A has two distinct eigenvalues, it is diagonalisable. The other two matrices have $\lambda = 1$ as their only eigenvalue, while

$$\mathcal{N}(B - I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{N}(C - I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

- This shows that C is a diagonalisable matrix, whereas B is not.

Example: Powers of a 2×2 matrix, page 1

- We compute the powers A^n of the matrix A in the case that
$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}.$$
- The eigenvalues of A are the roots of the characteristic polynomial
$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5).$$
- This gives $\lambda = 1, 5$ and one may easily check that
$$\mathcal{N}(A - I) = \operatorname{Span}\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 5I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$
- According to the general theory, we must then have
$$B = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad D = B^{-1}AB = \begin{bmatrix} 1 & \\ & 5 \end{bmatrix}$$
$$\Longrightarrow \quad D^n = B^{-1}A^nB = \begin{bmatrix} 1 & \\ & 5^n \end{bmatrix}.$$

Example: Powers of a 2×2 matrix, page 2

- Solving the last equation for A^n, we now find that

$$A^n = BD^nB^{-1} = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 5^n \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} -3 & 5^n \\ 1 & 5^n \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}.$$

- Since the inverse of B is given by the formula

$$\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = -\frac{1}{4} \begin{bmatrix} 1 & -1 \\ -1 & -3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix},$$

we may combine the last two equations to conclude that

$$A^n = \frac{1}{4} \begin{bmatrix} -3 & 5^n \\ 1 & 5^n \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$$
$$= \frac{1}{4} \begin{bmatrix} 5^n + 3 & 3 \cdot 5^n - 3 \\ 5^n - 1 & 3 \cdot 5^n + 1 \end{bmatrix}.$$