Linear algebra II
Tutorial solutions #1

1. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}.$$

Since $\operatorname{tr} A = 8$ and $\det A = 15 - 8 = 7$, the characteristic polynomial is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 8\lambda + 7 = (\lambda - 1)(\lambda - 7).$$

The eigenvectors with eigenvalue $\lambda = 1$ satisfy the system $A\boldsymbol{v} = \boldsymbol{v}$, namely

$$(A - I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad 4x + 2y = 0 \quad \Longrightarrow \quad y = -2x.$$

This means that every eigenvector with eigenvalue $\lambda = 1$ must have the form

$$\boldsymbol{v} = \begin{bmatrix} x \\ -2x \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \qquad x \neq 0.$$

Similarly, the eigenvectors with eigenvalue $\lambda = 7$ are solutions of $A\boldsymbol{v} = 7\boldsymbol{v}$, so

$$(A - 7I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad 2x - 2y = 0 \quad \Longrightarrow \quad y = x$$

and every eigenvector with eigenvalue $\lambda = 7$ must have the form

$$\boldsymbol{v} = \begin{bmatrix} x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad x \neq 0.$$

2. Is the following matrix diagonalisable? Why or why not?

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}.$$

Since $\operatorname{tr} A = 6$ and $\det A = 9$, the characteristic polynomial of A is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,$$

so the only eigenvalue is $\lambda = 3$. The eigenvectors satisfy the system $(A - 3I)\boldsymbol{v} = 0$, namely

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad x + y = 0 \quad \Longrightarrow \quad \boldsymbol{v} = \begin{bmatrix} x \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad x \neq 0.$$

Since A has only one linearly independent eigenvector, it is not diagonalisable.

3. Find a matrix A that has $\boldsymbol{v}_1$ as an eigenvector with eigenvalue $\lambda_1 = 2$ and $\boldsymbol{v}_2$ as an eigenvector with eigenvalue $\lambda_2 = 5$ when

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If B is the matrix whose columns are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, then the general theory implies that

$$B^{-1}AB = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

Once we now solve this equation for A, we may conclude that

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 4 & 5 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}.$$

4. Two square matrices A, C are said to be similar, if $C = B^{-1}AB$ for some invertible matrix B. Show that similar matrices have the same characteristic polynomial and also the same eigenvalues. Hint: one has $C - \lambda I = B^{-1}(A - \lambda I)B$.

Using the identity in the hint and properties of the determinant, we get

$$\det(C - \lambda I) = \det B^{-1} \cdot \det(A - \lambda I) \cdot \det B = \det(A - \lambda I).$$

This shows that A, C have the same characteristic polynomial. The eigenvalues are merely the roots of this polynomial, so A, C have the same eigenvalues as well.

Linear algebra II
Tutorial solutions #2

1. Find a basis for both the null space and the column space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 0 & 2 & 6 \\ 1 & 1 & 3 & 4 \end{bmatrix}.$$

The reduced row echelon form of A is easily found to be

$$R = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the pivots appear in the first and the second columns, this implies that

$$\mathcal{C}(A) = \text{Span}\{A\boldsymbol{e}_1, A\boldsymbol{e}_2\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The null space of A is the same as the null space of R. It can be expressed in the form

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -x_3 - 3x_4 \\ -2x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2. Find the eigenvalues and the generalised eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -6 & 3 \\ 0 & -1 & 4 \\ 1 & -2 & 2 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = (3 - \lambda)(\lambda - 1)^2.$$

When it comes to the eigenvalue $\lambda = 3$, one can easily check that

$$\mathcal{N}(A - 3I) = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 3I)^2 = \mathcal{N}(A - 3I).$$

This implies that $\mathcal{N}(A-3I)^j = \mathcal{N}(A-3I)$ for all $j \geq 1$, so we have found all generalised eigenvectors with $\lambda = 3$. When it comes to the eigenvalue $\lambda = 1$, one similarly has

$$\mathcal{N}(A-I) = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A-I)^2 = \mathcal{N}(A-I)^3 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

In view of the general theory, we must thus have $\mathcal{N}(A-I)^j = \mathcal{N}(A-I)^2$ for all $j \geq 2$.

3. Find a relation between x, y, z such that the following matrix is diagonalisable.

$$A = \begin{bmatrix} 1 & x & y \\ 0 & 2 & z \\ 0 & 0 & 1 \end{bmatrix}.$$

Since A is upper triangular, its eigenvalues are its diagonal entries $\lambda = 1, 1, 2$. Let us now worry about the eigenvectors. When $\lambda = 2$, we can use row reduction to get

$$A - 2I = \begin{bmatrix} -1 & x & y \\ 0 & 0 & z \\ 0 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -x & -y \\ 0 & 0 & 1 \\ 0 & 0 & z \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -x & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives 2 pivots and 1 linearly independent eigenvector. When $\lambda = 1$, we get

$$A - I = \begin{bmatrix} 0 & x & y \\ 0 & 1 & z \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & z \\ 0 & x & y \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & z \\ 0 & 0 & y - xz \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives 2 pivots and 1 linearly independent eigenvector in the case that $y \neq xz$, but it gives 1 pivot and 2 linearly independent eigenvectors in the case that $y = xz$. Thus, A is diagonalisable if and only if $y = xz$.

4. Suppose that λ is an eigenvalue of a square matrix A and let j be a positive integer. Show that the null space of $(A - \lambda I)^j$ is an A-invariant subspace of $\mathbb{C}^n$.

Assuming that $\boldsymbol{v}$ is in the null space of $(A - \lambda I)^j$, one can easily check that

$$\begin{aligned} (A - \lambda I)^j A\boldsymbol{v} &= (A - \lambda I)^j (A - \lambda I + \lambda I)\boldsymbol{v} \\ &= (A - \lambda I)^{j+1}\boldsymbol{v} + \lambda(A - \lambda I)^j \boldsymbol{v} = 0. \end{aligned}$$

This means that $A\boldsymbol{v}$ is also in the null space, so the null space is A-invariant.

Linear algebra II
Tutorial solutions #3

1. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ -7 & 5 & 3 \\ -5 & 1 & 6 \end{bmatrix}.$$

The characteristic polynomial of the given matrix is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 10\lambda^2 - 33\lambda + 36 = (4 - \lambda)(\lambda - 3)^2.$$

Thus, the eigenvalues are $\lambda = 3, 3, 4$ and one can easily find the null spaces

$$\mathcal{N}(A - 3I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 4I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

This implies that A is not diagonalisable and that its Jordan form is

$$J = B^{-1}AB = \left[\begin{array}{cc|c} 3 & & \\ 1 & 3 & \\ \hline & & 4 \end{array}\right].$$

To find a Jordan basis, we need to find a Jordan chain $\boldsymbol{v}_1, \boldsymbol{v}_2$ with eigenvalue $\lambda = 3$ and an eigenvector $\boldsymbol{v}_3$ with eigenvalue $\lambda = 4$. In our case, we have

$$\mathcal{N}(A - 3I)^2 = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\},$$

so it easily follows that a Jordan basis is provided by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = (A - 3I)\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

2. Suppose that A is a 4×4 matrix whose column space is equal to its null space. Show that A^2 must be the zero matrix and find the Jordan form of A.

To show that A^2 is the zero matrix, we note that its columns are all zero, as

$$A\boldsymbol{e}_i \in \mathcal{C}(A) \implies A\boldsymbol{e}_i \in \mathcal{N}(A) \implies A(A\boldsymbol{e}_i) = 0 \implies A^2\boldsymbol{e}_i = 0.$$

To find the Jordan form of A, we note that the null space of A is two-dimensional since

$$\dim \mathcal{C}(A) + \dim \mathcal{N}(A) = 4 \quad\Longrightarrow\quad 2\dim \mathcal{N}(A) = 4 \quad\Longrightarrow\quad \dim \mathcal{N}(A) = 2.$$

Also, the null space of A^2 is four-dimensional because A^2 is the zero matrix by above. This gives the Jordan chain diagram for the eigenvalue $\lambda = 0$, so the Jordan form is

$$J = \left[\begin{array}{cc|cc} 0 & & & \\ 1 & 0 & & \\ \hline & & 0 & \\ & & 1 & 0 \end{array}\right].$$

3. Suppose that A is a 4×4 matrix whose only eigenvalue is $\lambda = 1$. Suppose also that the column space of $A - I$ is one-dimensional. Find the Jordan form of A.

The eigenvalue $\lambda = 1$ has multiplicity 4 and the number of Jordan chains is

$$\dim \mathcal{N}(A - I) = 4 - \dim \mathcal{C}(A - I) = 3.$$

In particular, the Jordan chain diagram is and the Jordan form is

$$J = \left[\begin{array}{cc|c|c} 1 & & & \\ 1 & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \end{array}\right].$$

4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ form a basis for the null space of a square matrix A and that $C = B^{-1}AB$ for some invertible matrix B. Find a basis for the null space of C.

We note that a vector $\boldsymbol{v}$ lies in the null space of $C = B^{-1}AB$ if and only if

$$B^{-1}AB\boldsymbol{v} = 0 \quad\Longleftrightarrow\quad AB\boldsymbol{v} = 0 \quad\Longleftrightarrow\quad B\boldsymbol{v} \in \mathcal{N}(A) \quad\Longleftrightarrow\quad B\boldsymbol{v} = \sum_{i=1}^{k} c_i \boldsymbol{v}_i$$

for some scalar coefficients c_i. In particular, the vectors $B^{-1}\boldsymbol{v}_i$ span the null space of C. To show that these vectors are also linearly independent, we note that

$$\sum_{i=1}^{k} c_i B^{-1}\boldsymbol{v}_i = 0 \quad\Longleftrightarrow\quad \sum_{i=1}^{k} c_i \boldsymbol{v}_i = 0 \quad\Longleftrightarrow\quad c_i = 0 \text{ for all } i.$$

Linear algebra II
Tutorial solutions #4

1. The following matrix has $\lambda = 1$ as a triple eigenvalue. Use this fact to find its Jordan form, its minimal polynomial and also its inverse.

$$A = \begin{bmatrix} -1 & 4 & -4 \\ -2 & 5 & -4 \\ -1 & 2 & -1 \end{bmatrix}.$$

To find the Jordan form of A, we use row reduction on the matrix $A - \lambda I$ to get

$$A - \lambda I = A - I = \begin{bmatrix} -2 & 4 & -4 \\ -2 & 4 & -4 \\ -1 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives one pivot and two free variables, so there are two Jordan blocks and

$$J = \left[\begin{array}{cc|c} 1 & & \\ 1 & 1 & \\ \hline & & 1 \end{array}\right].$$

The minimal polynomial is $m(\lambda) = (\lambda - 1)^2$. Since A satisfies this polynomial, one has

$$A^2 - 2A + I = 0 \quad\Longrightarrow\quad I = A(2I - A) \quad\Longrightarrow\quad A^{-1} = 2I - A = \begin{bmatrix} 3 & -4 & 4 \\ 2 & -3 & 4 \\ 1 & -2 & 3 \end{bmatrix}.$$

2. Suppose that A is a 5×5 matrix whose minimal polynomial is $m(\lambda) = \lambda^2(\lambda - 1)$ and whose null space is 3-dimensional. Find the Jordan form of A.

Since the null space is 3-dimensional, there are 3 Jordan blocks with $\lambda = 0$. The largest of those is a 2×2 block, so $\lambda = 0$ has multiplicity at least 4. Since A is a 5×5 matrix that also has $\lambda = 1$ as an eigenvalue, $\lambda = 0$ has multiplicity exactly 4 and the Jordan form is

$$J = \left[\begin{array}{cc|c|c|c} 0 & & & & \\ 1 & 0 & & & \\ \hline & & 0 & & \\ \hline & & & 0 & \\ \hline & & & & 1 \end{array}\right].$$

3. Let P_1 be the space of all real polynomials of degree at most 1 and define

$$\langle f, g\rangle = \int_0^1 (x+1) f(x) g(x)\, dx \quad \text{for all } f, g \in P_1.$$

Find the matrix A of this bilinear form with respect to the standard basis.

The standard basis consists of the polynomials $\boldsymbol{v}_1 = 1$ and $\boldsymbol{v}_2 = x$, so

$$\langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = \langle x^{i-1}, x^{j-1}\rangle = \int_0^1 (x+1)\cdot x^{i+j-2}\, dx = \frac{1}{i+j} + \frac{1}{i+j-1}$$

for all integers $1 \leq i, j \leq 2$. Since the matrix A has entries $a_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle$, one finds that

$$A = \begin{bmatrix} 3/2 & 5/6 \\ 5/6 & 7/12 \end{bmatrix}.$$

4. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 4x_1y_1 + 2x_1y_2 + 2x_2y_1 + 7x_2y_2.$$

Find the matrix A of this form with respect to the standard basis and then find the matrix with respect to a basis consisting of eigenvectors of A.

By definition, the (i,j)th entry of A is the coefficient of x_iy_j and this means that

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}.$$

The characteristic polynomial of A and its eigenvalues are

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 11\lambda + 24 \quad \Longrightarrow \quad \lambda_1 = 3, \qquad \lambda_2 = 8,$$

while the corresponding eigenvectors are easily found to be

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Finally, the matrix of the form with respect to the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ has entries

$$b_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle = \boldsymbol{v}_i^t A \boldsymbol{v}_j = \lambda_j \boldsymbol{v}_i^t \boldsymbol{v}_j \quad \Longrightarrow \quad B = \begin{bmatrix} 15 & 0 \\ 0 & 40 \end{bmatrix}.$$

Linear algebra II
Tutorial solutions #5

1. Find an orthogonal matrix B such that B^tAB is diagonal when

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 9\lambda + 14 = (\lambda - 2)(\lambda - 7),$$

namely $\lambda_1 = 2$ and $\lambda_2 = 7$. It is easy to check that the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Dividing each of those by its length, one obtains an orthogonal matrix B such that

$$B = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \quad \Longrightarrow \quad B^tAB = B^{-1}AB = \begin{bmatrix} 2 & \\ & 7 \end{bmatrix}.$$

2. A real matrix A is called skew-symmetric, if $A^t = -A$. Show that the eigenvalues of such a matrix are purely imaginary, namely of the form $\lambda = iy$ with $y \in \mathbb{R}$.

Assuming that $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ, one finds that

$$\lambda \langle \boldsymbol{v}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \lambda \boldsymbol{v} \rangle = \langle \boldsymbol{v}, A\boldsymbol{v} \rangle = \langle A^* \boldsymbol{v}, \boldsymbol{v} \rangle.$$

Since $A^* = A^t = -A$, conjugate linearity in the first variable now gives

$$\lambda \langle \boldsymbol{v}, \boldsymbol{v} \rangle = \langle A^* \boldsymbol{v}, \boldsymbol{v} \rangle = \langle -A\boldsymbol{v}, \boldsymbol{v} \rangle = \langle -\lambda \boldsymbol{v}, \boldsymbol{v} \rangle = -\overline{\lambda} \langle \boldsymbol{v}, \boldsymbol{v} \rangle.$$

Writing $\lambda = x + iy$ for some real numbers x and y, we conclude that

$$\lambda = -\overline{\lambda} \quad \Longrightarrow \quad x + iy = -x + iy \quad \Longrightarrow \quad x = 0 \quad \Longrightarrow \quad \lambda = iy.$$

3. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form an orthonormal basis of $\mathbb{R}^n$ and consider the $n \times n$ matrix $A = I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t$. Show that $A^2 = I_n$ and find the Jordan form of A.

When it comes to the first part, it is easy to check that

$$A^2 = (I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t)(I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t) = I_n - 4\boldsymbol{v}_1\boldsymbol{v}_1^t + 4\boldsymbol{v}_1(\boldsymbol{v}_1^t\boldsymbol{v}_1)\boldsymbol{v}_1^t = I_n.$$

When it comes to the second part, the given vectors are all eigenvectors because

$$A\boldsymbol{v}_1 = \boldsymbol{v}_1 - 2\boldsymbol{v}_1(\boldsymbol{v}_1^t\boldsymbol{v}_1) = -\boldsymbol{v}_1, \qquad A\boldsymbol{v}_k = \boldsymbol{v}_k - 2\boldsymbol{v}_1(\boldsymbol{v}_1^t\boldsymbol{v}_k) = \boldsymbol{v}_k$$

for each $k \geq 2$. This means that A has n linearly independent eigenvectors, so its Jordan form is diagonal and the diagonal entries are the eigenvalues $\lambda = -1, 1, 1, \ldots, 1$.

4. Find a 2×2 symmetric matrix A with eigenvalues $\lambda = 1, 2$ such that $\boldsymbol{v}_1 = \left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right]$ is an eigenvector of A with eigenvalue $\lambda_1 = 1$.

The eigenvalues are distinct, so the eigenvectors are orthogonal to one another and the second eigenvector is a scalar multiple of $\boldsymbol{v}_2 = \left[\begin{smallmatrix} -4 \\ 3 \end{smallmatrix}\right]$. Dividing each of the eigenvectors by its length, one obtains an orthogonal matrix B such that

$$B = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \quad \Longrightarrow \quad B^t AB = B^{-1}AB = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}.$$

Once we now solve this equation for A, we may finally conclude that

$$A = \frac{1}{25}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & \\ & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 41 & -12 \\ -12 & 34 \end{bmatrix}.$$

5. Find an orthonormal basis of $\mathbb{R}^3$ that consists entirely of eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 6\lambda^2 = \lambda^2(6 - \lambda),$$

namely $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 6$. The corresponding eigenvectors are easily found to be

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Note that the first two vectors are not orthogonal to one another. To find an orthogonal basis, we thus resort to the Gram-Schmidt procedure which gives the vectors

$$\boldsymbol{w}_1 = \boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle}\boldsymbol{w}_1 = \begin{bmatrix} -1/5 \\ -2/5 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Once we now divide each of these vectors by its length, we get the orthonormal basis

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \frac{1}{\sqrt{30}}\begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}, \qquad \boldsymbol{u}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Linear algebra II
Tutorial solutions #6

1. Find the values of a for which the matrix A is positive definite when

$$A = \begin{bmatrix} a & 1 & a \\ 1 & a & 1 \\ a & 1 & 2 \end{bmatrix}.$$

According to Sylvester's criterion, one needs to have

$$a > 0, \qquad a^2 - 1 > 0, \qquad \det A > 0.$$

The first two conditions require that $a > 1$, while the last condition requires

$$\det A = -a^3 + 2a^2 + a - 2 = -(a-2)(a-1)(a+1)$$

to be positive. It easily follows that A is positive definite if and only if $1 < a < 2$.

2. Prove the converse of the spectral theorem: if A is a real matrix such that B^tAB is diagonal for some orthogonal matrix B, then A must actually be symmetric.

Since the matrix B^tAB is diagonal, it is equal to its transpose and this implies that

$$B^tAB = (B^tAB)^t = B^tA^tB^{tt} \implies AB = A^tB \implies A = A^t.$$

3. Find all $n \times n$ real symmetric matrices A such that $A^3 = I_n$.

Since A is symmetric with $A^3 = I_n$, its eigenvalues are real with $\lambda^3 = 1$ and this means that every eigenvalue is equal to 1. In particular, there exists an orthogonal matrix B such that $B^tAB = B^{-1}AB$ is diagonal with diagonal entries equal to 1, so

$$B^{-1}AB = I_n \implies AB = B \implies A = I_n.$$

4. Show that $A = I_n + \boldsymbol{v}\boldsymbol{v}^t$ is positive definite symmetric for each vector $\boldsymbol{v} \in \mathbb{R}^n$.

To show that A is symmetric, we note that

$$A^t = I_n^t + (\boldsymbol{v}\boldsymbol{v}^t)^t = I_n + \boldsymbol{v}^{tt}\boldsymbol{v}^t = I_n + \boldsymbol{v}\boldsymbol{v}^t = A.$$

To show that A is positive definite, we let $\boldsymbol{x} \in \mathbb{R}^n$ be a nonzero vector and we note that

$$\boldsymbol{x}^tA\boldsymbol{x} = \boldsymbol{x}^t(I_n + \boldsymbol{v}\boldsymbol{v}^t)\boldsymbol{x} = \boldsymbol{x}^t\boldsymbol{x} + (\boldsymbol{x}^t\boldsymbol{v})(\boldsymbol{v}^t\boldsymbol{x}) = ||\boldsymbol{x}||^2 + (\boldsymbol{x} \cdot \boldsymbol{v})^2 > 0.$$

5. Find the values of a for which $Q(x,y,z)$ is positive definite when

$$Q(x,y,z) = x^2 + (a+2)y^2 + az^2 + 2axy + 2axz + 2yz.$$

The given quadratic can be expressed in the form $Q(\boldsymbol{x}) = \boldsymbol{x}^t A \boldsymbol{x}$, where

$$A = \begin{bmatrix} 1 & a & a \\ a & a+2 & 1 \\ a & 1 & a \end{bmatrix}.$$

According to Sylvester's criterion, this matrix is positive definite if and only if

$$\det \begin{bmatrix} 1 & a \\ a & a+2 \end{bmatrix} = a + 2 - a^2, \qquad \det A = -2a^3 + a^2 + 2a - 1$$

are both positive. When it comes to the first determinant, one finds that

$$a + 2 - a^2 > 0 \quad \Longleftrightarrow \quad (a+1)(a-2) < 0 \quad \Longleftrightarrow \quad -1 < a < 2.$$

When it comes to the second determinant, one similarly finds that

$$\det A = -2a^3 + a^2 + 2a - 1 = -(a+1)(a-1)(2a-1).$$

It easily follows that A is positive definite if and only if $1/2 < a < 1$.

6. Suppose that A is a real positive definite symmetric matrix. Show that there exists a real positive definite symmetric matrix Q such that $Q^2 = A$.

Since A is positive definite symmetric, it is diagonalisable with positive eigenvalues λ_i. In fact, there exists a real orthogonal matrix B such that

$$B^t AB = B^{-1}AB = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

Let us denote by D the diagonal matrix with diagonal entries $\sqrt{\lambda_i}$. Then $D^2 = B^t AB$ and

$$(BDB^t)^2 = BD(B^t B)DB^t = BD^2 B^t = BB^t ABB^t = A.$$

In particular, the matrix $Q = BDB^t$ satisfies the given condition. This matrix is symmetric because $Q^t = B^{tt} D^t B^t = BDB^t = Q$ and it is positive definite because its eigenvalues are the same as the eigenvalues of D, so they are all positive.