Linear algebra II
Homework #1 solutions

1. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}.$$

Since $\operatorname{tr} A = 9$ and $\det A = 18 + 2 = 20$, the characteristic polynomial is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 9\lambda + 20 = (\lambda - 4)(\lambda - 5).$$

The eigenvectors with eigenvalue $\lambda = 4$ satisfy the system $A\boldsymbol{v} = 4\boldsymbol{v}$, namely

$$(A - 4I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad -x + 2y = 0 \quad \Longrightarrow \quad x = 2y.$$

This means that every eigenvector with eigenvalue $\lambda = 4$ must have the form

$$\boldsymbol{v} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad y \neq 0.$$

Similarly, the eigenvectors with eigenvalue $\lambda = 5$ are solutions of $A\boldsymbol{v} = 5\boldsymbol{v}$, so

$$(A - 5I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad -x + y = 0 \quad \Longrightarrow \quad x = y$$

and every eigenvector with eigenvalue $\lambda = 5$ must have the form

$$\boldsymbol{v} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad y \neq 0.$$

2. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 2 & 2 & -1 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = -(\lambda - 1)^2(\lambda - 3).$$

The eigenvectors of A are nonzero vectors in the null spaces

$$\mathcal{N}(A - I) = \operatorname{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 3I) = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

3. The following matrix has eigenvalues $\lambda = 1, 1, 2, 2$. Is it diagonalisable? Explain.

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 2 & -1 & 1 \\ -1 & 1 & 0 & 2 \\ -2 & 2 & -1 & 3 \end{bmatrix}.$$

When it comes to the eigenvalue $\lambda = 1$, row reduction of $A - \lambda I$ gives

$$A - \lambda I = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ -2 & 2 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so there are 3 pivots and only 1 linearly independent eigenvector. When $\lambda = 2$, we have

$$A - \lambda I = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & -2 & 2 \\ -2 & 2 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so we get 2 pivots and 2 linearly independent eigenvectors. This gives a total of 3 linearly independent eigenvectors, so A is not diagonalisable. One does not really need to find the eigenvectors in this case, but those are nonzero elements of the null spaces

$$\mathcal{N}(A - I) = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 2I) = \operatorname{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

4. Suppose A is a diagonalisable matrix and let $k \geq 1$ be an integer. Show that each eigenvector of A is an eigenvector of A^k and conclude that A^k is diagonalisable.

If $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ, then $\boldsymbol{v}$ satisfies $A\boldsymbol{v} = \lambda\boldsymbol{v}$, so

$$A^2\boldsymbol{v} = A(A\boldsymbol{v}) = A(\lambda\boldsymbol{v}) = \lambda(A\boldsymbol{v}) = \lambda(\lambda\boldsymbol{v}) = \lambda^2\boldsymbol{v}.$$

It easily follows by induction that $A^k\boldsymbol{v} = \lambda^k\boldsymbol{v}$ for each k. In particular, $\boldsymbol{v}$ is an eigenvector of A^k as well. Suppose that A is $n \times n$. Being diagonalisable, it must then have n linearly independent eigenvectors. Those are also eigenvectors of A^k, so this matrix has n linearly independent eigenvectors and it is diagonalisable as well.

Linear algebra II
Homework #2 solutions

1. Let $x_0 = 3$ and $y_0 = 1$. Suppose the sequences x_n, y_n are such that

$$x_n = 3x_{n-1} - 2y_{n-1}, \qquad y_n = 4x_{n-1} + 9y_{n-1}$$

for each $n \geq 1$. Determine each of x_n and y_n explicitly in terms of n.

Letting $\boldsymbol{u}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -2 \\ 4 & 9 \end{bmatrix}$, one easily finds that

$$\boldsymbol{u}_n = A\boldsymbol{u}_{n-1} = A^2\boldsymbol{u}_{n-2} = \ldots = A^n\boldsymbol{u}_0.$$

In particular, it remains to compute A^n. The eigenvalues of A are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 12\lambda + 35 = 0 \quad \Longrightarrow \quad \lambda = 5, 7$$

and we may proceed as usual to obtain the corresponding eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Let B be the matrix whose columns are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Then $B^{-1}AB$ is diagonal and

$$B^{-1}AB = \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \quad \Longrightarrow \quad B^{-1}A^nB = \begin{bmatrix} 5^n & 0 \\ 0 & 7^n \end{bmatrix}.$$

Solving this equation for A^n and simplifying, we now get

$$A^n = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^n & 0 \\ 0 & 7^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 5^n - 7^n & 5^n - 7^n \\ 2 \cdot 7^n - 2 \cdot 5^n & 2 \cdot 7^n - 5^n \end{bmatrix}.$$

In particular, the sequences x_n, y_n are given explicitly by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \boldsymbol{u}_n = A^n\boldsymbol{u}_0 = \begin{bmatrix} 7 \cdot 5^n - 4 \cdot 7^n \\ 8 \cdot 7^n - 7 \cdot 5^n \end{bmatrix}.$$

2. Show that the following matrix is diagonalisable.

$$A = \begin{bmatrix} 7 & 1 & -7 \\ 3 & 3 & -5 \\ 3 & 1 & -3 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 14\lambda + 8 = -(\lambda - 1)(\lambda - 2)(\lambda - 4).$$

Since the eigenvalues of A are distinct, we conclude that A is diagonalisable.

3. Find the eigenvalues and the generalised eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = (3 - \lambda)(\lambda - 2)^2.$$

When it comes to the eigenvalue $\lambda = 3$, one can easily check that

$$\mathcal{N}(A - 3I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 3I)^2 = \mathcal{N}(A - 3I).$$

This implies that $\mathcal{N}(A - 3I)^j = \mathcal{N}(A - 3I)$ for all $j \geq 1$, so we have found all generalised eigenvectors with $\lambda = 3$. When it comes to the eigenvalue $\lambda = 2$, one similarly has

$$\mathcal{N}(A - 2I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 2I)^2 = \mathcal{N}(A - 2I)^3 = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

In view of the general theory, we must thus have $\mathcal{N}(A - 2I)^j = \mathcal{N}(A - 2I)^2$ for all $j \geq 2$.

4. Suppose that A is a 4×4 matrix whose first two columns are linearly independent, its third column is equal to the first column and its last column is zero. Find a basis for both the column space and the null space of A. Hint: one has $A\boldsymbol{e}_3 = A\boldsymbol{e}_1$ and $A\boldsymbol{e}_4 = 0$.

By assumption, the first two columns are linearly independent, while the other two columns are linear combinations of the first two. This implies that $A\boldsymbol{e}_1$, $A\boldsymbol{e}_2$ form a basis for the column space. Since the column space is two-dimensional, the null space must be two-dimensional as well. On the other hand, the given assumptions ensure that

$$A(\boldsymbol{e}_3 - \boldsymbol{e}_1) = A\boldsymbol{e}_3 - A\boldsymbol{e}_1 = 0, \qquad A\boldsymbol{e}_4 = 0.$$

It easily follows that the vectors $\boldsymbol{e}_3 - \boldsymbol{e}_1$ and $\boldsymbol{e}_4$ form a basis for the null space, namely

$$\mathcal{N}(A) = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Linear algebra II
Homework #3 solutions

1. The following matrix has $\lambda = 2$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}.$$

In this case, the null space of $A - 2I$ is two-dimensional, as row reduction gives

$$A - 2I = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, $(A-2I)^2$ is the zero matrix, so its null space is three-dimensional. Thus, the diagram of Jordan chains is and there is a Jordan chain of length 2 as well as a Jordan chain of length 1. These Jordan chains give a 2×2 block and an 1×1 block, so

$$J = \left[\begin{array}{cc|c} 2 & & \\ 1 & 2 & \\ \hline & & 2 \end{array}\right].$$

2. The following matrix has $\lambda = 2$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & 6 & 1 \end{bmatrix}.$$

In this case, the null space of $A - 2I$ is one-dimensional, as row reduction gives

$$A - 2I = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & -1 \\ -1 & 6 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -4 & 1 \\ 0 & 8 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly, the null space of $(A - 2I)^2$ is two-dimensional because

$$(A - 2I)^2 = \begin{bmatrix} 6 & -4 & -2 \\ 3 & -2 & -1 \\ 12 & -8 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2/3 & -1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

while $(A - 2I)^3$ is the zero matrix, so its null space is three-dimensional. The diagram of Jordan chains is then and there is a single 3×3 Jordan block, namely

$$J = \begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix} = \begin{bmatrix} 2 & & \\ 1 & 2 & \\ & 1 & 2 \end{bmatrix}.$$

3. Find a Jordan chain of length 2 for the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}.$$

The eigenvalues of the given matrix are the roots of the characteristic polynomial

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,$$

so $\lambda = 3$ is the only eigenvalue. Using row reduction, we now get

$$A - 3I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix},$$

so the null space of $A - 3I$ is one-dimensional. On the other hand, $(A - 3I)^2$ is the zero matrix, so its null space is two-dimensional. To find a Jordan chain of length 2, we pick a vector $\boldsymbol{v}_1$ that lies in the latter null space, but not in the former. We can always take

$$\boldsymbol{v}_1 = \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{v}_2 = (A - 3I)\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix},$$

but there are obviously infinitely many choices. Another possible choice would be

$$\boldsymbol{v}_1 = \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{v}_2 = (A - 3I)\boldsymbol{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

4. Let $\boldsymbol{x} \in \mathbb{R}^3$ be nonzero and let A be the matrix whose columns are $\boldsymbol{x}, 2\boldsymbol{x}, 3\boldsymbol{x}$ in this order. Show that $\boldsymbol{x}$ is an eigenvector of A and find a basis for the null space of A.

The columns of A are $A\boldsymbol{e}_1 = \boldsymbol{x}$, $A\boldsymbol{e}_2 = 2\boldsymbol{x}$ and $A\boldsymbol{e}_3 = 3\boldsymbol{x}$. It easily follows that

$$A\boldsymbol{x} = A(x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3) = x_1\boldsymbol{x} + x_2(2\boldsymbol{x}) + x_3(3\boldsymbol{x}) = \lambda\boldsymbol{x},$$

where $\lambda = x_1 + 2x_2 + 3x_3$. This shows that $\boldsymbol{x}$ is an eigenvector of A. Since every column of A is a scalar multiple of $\boldsymbol{x}$, the column space is one-dimensional and the null space is two-dimensional. Using the condition $2A\boldsymbol{e}_1 = 2\boldsymbol{x} = A\boldsymbol{e}_2$, one finds that $2\boldsymbol{e}_1 - \boldsymbol{e}_2 \in \mathcal{N}(A)$. Using the condition $3A\boldsymbol{e}_1 = 3\boldsymbol{x} = A\boldsymbol{e}_3$, we get $3\boldsymbol{e}_1 - \boldsymbol{e}_3 \in \mathcal{N}(A)$ as well, hence

$$\mathcal{N}(A) = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Linear algebra II
Homework #4 solutions

1. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & -2 \\ 4 & 8 & -3 \end{bmatrix}.$$

The characteristic polynomial of the given matrix is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = (3 - \lambda)(\lambda - 1)^2,$$

so its eigenvalues are $\lambda = 1, 1, 3$. The corresponding null spaces are easily found to be

$$\mathcal{N}(A - I) = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 3I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

These contain 3 linearly independent eigenvectors, so A is diagonalisable and

$$B = \begin{bmatrix} -2 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \Longrightarrow \quad J = B^{-1}AB = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 3 \end{bmatrix}.$$

2. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 4 & -1 \\ 1 & 3 & 1 \end{bmatrix}.$$

The characteristic polynomial of the given matrix is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 8\lambda^2 - 21\lambda + 18 = (2 - \lambda)(\lambda - 3)^2,$$

so its eigenvalues are $\lambda = 2, 3, 3$. The corresponding null spaces are easily found to be

$$\mathcal{N}(A - 2I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 3I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

This implies that A is not diagonalisable and that its Jordan form is

$$J = B^{-1}AB = \left[\begin{array}{c|cc} 2 & & \\ \hline & 3 & \\ & 1 & 3 \end{array}\right].$$

To find a Jordan basis, we need to find vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ such that $\boldsymbol{v}_1$ is an eigenvector with eigenvalue $\lambda = 2$ and $\boldsymbol{v}_2, \boldsymbol{v}_3$ is a Jordan chain with eigenvalue $\lambda = 3$. In our case, we have

$$\mathcal{N}(A-3I)^2 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

so it easily follows that a Jordan basis is provided by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = (A-3I)\boldsymbol{v}_2 = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}.$$

3. Suppose A is a 2×2 matrix such that $A^2 = I_2$ and let J be the Jordan form of A. Show that $J^2 = I_2$ and use this fact to conclude that J is diagonal.

Write $J = B^{-1}AB$ for some invertible matrix B. Since $A^2 = I_2$, we must also have

$$J^2 = B^{-1}AB \cdot B^{-1}AB = B^{-1}A^2B = B^{-1}I_2B = I_2.$$

Next, we show that J is diagonal. If that is not the case, then J is a 2×2 block and

$$J = \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix} \quad \Longrightarrow \quad J^2 = \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & \\ 2\lambda & \lambda^2 \end{bmatrix}.$$

Comparing the last two equations now gives $\lambda^2 = 1$ and $2\lambda = 0$, a contradiction.

4. Suppose A is a 4×4 matrix with characteristic polynomial $f(\lambda) = \lambda^3(\lambda - 1)$ and suppose its column space is two-dimensional. Find the Jordan form of A.

When it comes to the triple eigenvalue $\lambda = 0$, the number of Jordan blocks is

$$\dim \mathcal{N}(A - \lambda I) = \dim \mathcal{N}(A) = 4 - \dim \mathcal{C}(A) = 2.$$

In particular, there is one 2×2 block and one 1×1 block, so the Jordan form is

$$J = \left[\begin{array}{cc|c|c} 0 & & & \\ 1 & 0 & & \\ \hline & & 0 & \\ \hline & & & 1 \end{array}\right].$$

Linear algebra II
Homework #5 solutions

1. Let $x_0 = 1$ and $y_0 = 2$. Suppose the sequences x_n, y_n are such that

$$x_n = 8x_{n-1} - 9y_{n-1}, \qquad y_n = x_{n-1} + 2y_{n-1}$$

for each $n \geq 1$. Determine each of x_n and y_n explicitly in terms of n.

As we already know, one may express this problem in terms of matrices by writing

$$\boldsymbol{u}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 8 & -9 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = A\boldsymbol{u}_{n-1} \quad \Longrightarrow \quad \boldsymbol{u}_n = A^n \boldsymbol{u}_0.$$

Let us now focus on computing A^n. The characteristic polynomial of A is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2,$$

so the only eigenvalue is $\lambda = 5$. The eigenvectors of A are nonzero elements of the null space

$$\mathcal{N}(A - 5I) = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\},$$

while $(A - 5I)^2$ is the zero matrix. Thus, a Jordan basis is provided by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = (A - 5I)\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Letting B be the matrix whose columns are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we must thus have

$$J = B^{-1}AB = \begin{bmatrix} 5 & \\ 1 & 5 \end{bmatrix} \quad \Longrightarrow \quad J^n = B^{-1}A^nB = \begin{bmatrix} 5^n & \\ n5^{n-1} & 5^n \end{bmatrix}.$$

Once we now solve this equation for A^n, we find that

$$A^n = BJ^nB^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5^n & \\ n5^{n-1} & 5^n \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (5+3n)5^{n-1} & -9n5^{n-1} \\ n5^{n-1} & (5-3n)5^{n-1} \end{bmatrix}.$$

In particular, the sequences x_n, y_n are given explicitly by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \boldsymbol{u}_n = A^n\boldsymbol{u}_0 = \begin{bmatrix} (1-3n)5^n \\ (2-n)5^n \end{bmatrix}.$$

2. Which of the following matrices are similar? Explain.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

The eigenvalues of the first matrix are given by

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \quad \Longrightarrow \quad \lambda^2 - 4\lambda + 3 = 0 \quad \Longrightarrow \quad \lambda = 1, 3$$

and the eigenvalues of the second matrix are

$$\lambda^2 - (\operatorname{tr} B)\lambda + \det B = 0 \quad \Longrightarrow \quad \lambda^2 - 4\lambda + 4 = 0 \quad \Longrightarrow \quad \lambda = 2, 2.$$

The other two matrices are lower triangular, so their eigenvalues are their diagonal entries. This means that B, C are the only two matrices which could be similar. In fact,

$$\mathcal{N}(B - 2I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

is one-dimensional, so the Jordan form of B has a single block and B is similar to C.

3. Show that the trace of a square matrix A is the sum of its eigenvalues. Hint: prove the same statement for the Jordan form of A and then use similarity.

Let J denote the Jordan form of A. Since J is lower triangular, its eigenvalues are its diagonal entries, so the sum of its eigenvalues is equal to its trace. On the other hand, A is similar to J, so the two matrices have both the same trace and the same eigenvalues. This means that the sum of the eigenvalues of A is equal to the trace of A.

4. Let $\boldsymbol{x} \in \mathbb{R}^3$ be nonzero and let A be the matrix whose columns are $\boldsymbol{x}, 2\boldsymbol{x}, 3\boldsymbol{x}$ in this order. Find the Jordan form of A. Hint: the answer depends on the trace of A; show that the null space is two-dimensional and that the eigenvalues are $\lambda = 0, 0, \operatorname{tr} A$.

Since the column space is one-dimensional, the null space must be two-dimensional. This means that the Jordan form contains two blocks with eigenvalue $\lambda = 0$. As the sum of the eigenvalues is equal to the trace, the third eigenvalue is thus $\lambda = \operatorname{tr} A$.

Let us now consider two cases. When $\operatorname{tr} A \neq 0$, the eigenvalue $\lambda = \operatorname{tr} A$ is simple and it contributes a single 1×1 block. There are also two Jordan blocks with $\lambda = 0$, so all blocks are 1×1 blocks and the Jordan form is diagonal. When $\operatorname{tr} A = 0$, the eigenvalue $\lambda = 0$ is a triple eigenvalue that only contributes two Jordan blocks, so the Jordan form is

$$J = \left[\begin{array}{cc|c} 0 & & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right].$$

Linear algebra II
Homework #6 solutions

1. The following matrix has $\lambda = 1$ as a triple eigenvalue. Use this fact to find its Jordan form, its minimal polynomial and also its inverse.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}.$$

To find the Jordan form of A, we use row reduction on the matrix $A - \lambda I$ to get

$$A - \lambda I = A - I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives one pivot and two free variables, so there are two Jordan blocks and

$$J = \left[\begin{array}{cc|c} 1 & & \\ 1 & 1 & \\ \hline & & 1 \end{array}\right].$$

The minimal polynomial is $m(\lambda) = (\lambda - 1)^2$. Since A satisfies this polynomial, one has

$$A^2 - 2A + I = 0 \quad \Longrightarrow \quad I = A(2I - A) \quad \Longrightarrow \quad A^{-1} = 2I - A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -2 & -2 & 3 \end{bmatrix}.$$

2. The following matrix has eigenvalues $\lambda = 0, 1, 1$. Use this fact to find its Jordan form, its minimal polynomial and also its power A^{2018}.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}.$$

The eigenvalue $\lambda = 0$ is simple, so it contributes an 1×1 block. When $\lambda = 1$, we have

$$A - \lambda I = A - I = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 3 & 1 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

and this gives a single 2×2 block with eigenvalue $\lambda = 1$. The Jordan form is thus

$$J = \left[\begin{array}{cc|c} 1 & & \\ 1 & 1 & \\ \hline & & 0 \end{array}\right]$$

and the minimal polynomial is $m(\lambda) = \lambda(\lambda - 1)^2$. Since A satisfies this polynomial,

$$\begin{aligned} A^3 - 2A^2 + A = 0 \quad &\Longrightarrow \quad A^3 = 2A^2 - A \\ &\Longrightarrow \quad A^4 = 2A^3 - A^2 = 2(2A^2 - A) - A^2 = 3A^2 - 2A \\ &\Longrightarrow \quad A^5 = 3A^3 - 2A^2 = 3(2A^2 - A) - 2A^2 = 4A^2 - 3A. \end{aligned}$$

It follows by induction that $A^n = (n-1)A^2 - (n-2)A$ for each $n \geq 3$ and this gives

$$A^{2018} = 2017 \begin{bmatrix} 3 & 2 & -2 \\ 3 & 2 & -2 \\ 5 & 3 & -3 \end{bmatrix} - 2016 \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2019 & 2018 & -2018 \\ 2019 & 2018 & -2018 \\ 4037 & 4035 & -4035 \end{bmatrix}.$$

3. Suppose that A is a 2×2 matrix with $\det A = 0$. Use the Cayley-Hamilton theorem to show that $A^2 = (\operatorname{tr} A)A$ and determine A^n explicitly for each integer $n \geq 2$.

According to the Cayley-Hamilton theorem, A satisfies its characteristic polynomial

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A \quad \Longrightarrow \quad A^2 = (\operatorname{tr} A)A - (\det A)I = (\operatorname{tr} A)A.$$

Using this formula to compute A^3, one now finds that

$$A^3 = A^2 \cdot A = (\operatorname{tr} A)A^2 = (\operatorname{tr} A)^2 A.$$

It easily follows by induction that $A^n = (\operatorname{tr} A)^{n-1} A$ for each integer $n \geq 2$.

4. Let P_2 be the space of all real polynomials of degree at most 2 and let

$$\langle f, g \rangle = \int_0^1 (1-x) \cdot f(x)g(x)\, dx \quad \text{for all } f, g \in P_2.$$

Find the matrix of this bilinear form with respect to the standard basis.

The standard basis consists of the polynomials $\boldsymbol{v}_1 = 1$, $\boldsymbol{v}_2 = x$, $\boldsymbol{v}_3 = x^2$ and this gives

$$\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \langle x^{i-1}, x^{j-1} \rangle = \int_0^1 (1-x) \cdot x^{i+j-2}\, dx = \frac{1}{i+j-1} - \frac{1}{i+j}$$

for all integers $1 \leq i, j \leq 3$. Since the matrix of the form has entries $a_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$, we get

$$A = \begin{bmatrix} 1/2 & 1/6 & 1/12 \\ 1/6 & 1/12 & 1/20 \\ 1/12 & 1/20 & 1/30 \end{bmatrix}.$$

Linear algebra II
Homework #7 solutions

1. Consider $\mathbb{R}^3$ with the usual dot product. Use the Gram-Schmidt procedure to find an orthogonal basis, starting with the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Keep the first vector and let $\boldsymbol{w}_1 = \boldsymbol{v}_1$. The second vector $\boldsymbol{v}_2$ must be replaced by

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1+0+3}{1+4+9} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/7 \\ -4/7 \\ 1/7 \end{bmatrix}$$

and then the third vector $\boldsymbol{v}_3$ must be replaced by

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \frac{\langle \boldsymbol{v}_3, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 - \frac{\langle \boldsymbol{v}_3, \boldsymbol{w}_2 \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle} \boldsymbol{w}_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ -1/3 \end{bmatrix}.$$

2. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_1 y_2 + x_2 y_1 + 5 x_2 y_2.$$

Show that this is an inner product and use the Gram-Schmidt procedure to find an orthogonal basis for it, starting with the standard basis of $\mathbb{R}^2$.

The given form is symmetric because its matrix with respect to the standard basis is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}.$$

To show that the form is positive definite, we complete the square to write

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = x_1^2 + 2x_1 x_2 + 5x_2^2 = (x_1 + x_2)^2 + 4x_2^2.$$

Finally, one can obtain an orthogonal basis by taking $\boldsymbol{w}_1 = \boldsymbol{e}_1$ and

$$\boldsymbol{w}_2 = \boldsymbol{e}_2 - \frac{\langle \boldsymbol{e}_2, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 = \boldsymbol{e}_2 - \frac{\boldsymbol{e}_2^t A \boldsymbol{e}_1}{\boldsymbol{e}_1^t A \boldsymbol{e}_1} \boldsymbol{e}_1 = \boldsymbol{e}_2 - \frac{a_{21}}{a_{11}} \boldsymbol{e}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

3. Define a bilinear form on the space M_{22} of all 2×2 real matrices by setting

$$\langle A, B \rangle = \operatorname{tr}(A^t B)$$

for all 2×2 real matrices A, B. Express this equation in terms of the entries of the two matrices. Is the bilinear form symmetric? Is it positive definite?

In terms of the entries of the two matrices, we have

$$A^t B = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{21} & a_{11}b_{12} + a_{21}b_{22} \\ a_{12}b_{11} + a_{22}b_{21} & a_{12}b_{12} + a_{22}b_{22} \end{bmatrix},$$

so one may write the given equation as

$$\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22} = \sum_{i,j} a_{ij}b_{ij}.$$

In particular, the given form is symmetric and we also have

$$\langle A, A \rangle = a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2 = \sum_{i,j} a_{ij}^2 \geq 0.$$

Since equality holds if and only if $A = 0$, the given form is positive definite as well.

4. Find two eigenvectors of A which form an orthonormal basis of $\mathbb{R}^2$ when

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \qquad a, b \neq 0.$$

First of all, we compute the eigenvalues of A. The characteristic polynomial is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 2a\lambda + a^2 - b^2 = (\lambda - a)^2 - b^2,$$

so it easily follows that the eigenvalues are

$$(\lambda - a)^2 = b^2 \quad \Longrightarrow \quad \lambda - a = \pm b \quad \Longrightarrow \quad \lambda = a \pm b.$$

Next, we turn to the eigenvectors. When $\lambda = a + b$, one may use row reduction to get

$$A - \lambda I = \begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{N}(A - \lambda I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

When $\lambda = a - b$, one may use a similar computation to find that

$$A - \lambda I = \begin{bmatrix} b & b \\ b & b \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathcal{N}(A - \lambda I) = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

This gives two vectors which are orthogonal to one another, so an orthonormal basis is

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Linear algebra II
Homework #8 solutions

1. Find an orthogonal matrix B such that B^tAB is diagonal when

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = -(\lambda - 1)(\lambda - 3)(\lambda - 6).$$

This gives $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = 6$, while the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since the eigenvalues are distinct, the eigenvectors are orthogonal to one another. Once we now divide each of them by its length, we obtain the columns of the orthogonal matrix

$$B = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

2. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form an orthonormal basis of $\mathbb{R}^n$ and consider the $n \times n$ matrix $A = \boldsymbol{v}_2\boldsymbol{v}_1^t$. Show that $A^2 = 0$ and find the Jordan form of A.

Since $\boldsymbol{v}_1$ is perpendicular to $\boldsymbol{v}_2$ by assumption, one easily finds that

$$\boldsymbol{v}_1^t\boldsymbol{v}_2 = 0 \quad \Longrightarrow \quad A^2 = \boldsymbol{v}_2\boldsymbol{v}_1^t\boldsymbol{v}_2\boldsymbol{v}_1^t = 0.$$

In particular, $\lambda = 0$ is the only eigenvalue of A and we also have

$$A\boldsymbol{v}_1 = \boldsymbol{v}_2\boldsymbol{v}_1^t\boldsymbol{v}_1 = \boldsymbol{v}_2, \qquad A\boldsymbol{v}_k = \boldsymbol{v}_2\boldsymbol{v}_1^t\boldsymbol{v}_k = 0$$

for each integer $k \geq 2$. Since $A\boldsymbol{v}_1 = \boldsymbol{v}_2$ and $A\boldsymbol{v}_2 = 0$, the first two vectors form a Jordan chain of length 2. Thus, the Jordan form contains a 2×2 block with eigenvalue $\lambda = 0$. The remaining blocks correspond to the vectors $\boldsymbol{v}_3, \ldots, \boldsymbol{v}_n$ and they are all 1×1 blocks.

3. Find an orthonormal basis of $\mathbb{R}^3$ that consists entirely of eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 14\lambda^2 = \lambda^2(14 - \lambda),$$

namely $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 14$. The corresponding eigenvectors are easily found to be

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Note that the first two vectors are not orthogonal to one another. To find an orthogonal basis, we thus resort to the Gram-Schmidt procedure which gives the vectors

$$\boldsymbol{w}_1 = \boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{w}_2 = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 = \begin{bmatrix} -3/5 \\ -6/5 \\ 1 \end{bmatrix}, \qquad \boldsymbol{w}_3 = \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Once we now divide each of these vectors by its length, we get the orthonormal basis

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \frac{1}{\sqrt{70}} \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}, \qquad \boldsymbol{u}_3 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

4. Find an orthogonal 3×3 matrix whose first two columns are

$$\boldsymbol{v}_1 = \begin{bmatrix} \cos x \cdot \cos y \\ \sin x \\ \cos x \cdot \sin y \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -\sin y \\ 0 \\ \cos y \end{bmatrix}, \qquad x, y \in \mathbb{R}.$$

It is easy to check that $\boldsymbol{v}_1, \boldsymbol{v}_2$ are orthogonal vectors of unit length, namely

$$\begin{aligned} \boldsymbol{v}_1^t \boldsymbol{v}_2 &= -\cos x \cdot \cos y \cdot \sin y + \cos x \cdot \cos y \cdot \sin y = 0, \\ ||\boldsymbol{v}_1||^2 &= \cos^2 x \cdot \cos^2 y + \cos^2 x \cdot \sin^2 y + \sin^2 x = \cos^2 x + \sin^2 x = 1, \\ ||\boldsymbol{v}_2||^2 &= \sin^2 y + \cos^2 y = 1. \end{aligned}$$

The third column is perpendicular to each of the first two columns, so it is parallel to

$$\boldsymbol{v}_3 = \boldsymbol{v}_1 \times \boldsymbol{v}_2 = \begin{bmatrix} \cos x \cdot \cos y \\ \sin x \\ \cos x \cdot \sin y \end{bmatrix} \times \begin{bmatrix} -\sin y \\ 0 \\ \cos y \end{bmatrix} = \begin{bmatrix} \sin x \cdot \cos y \\ -\cos x \\ \sin x \cdot \sin y \end{bmatrix}.$$

This is actually a unit vector itself, so the third column could be either $\boldsymbol{v}_3$ or $-\boldsymbol{v}_3$.