Linear algebra II
Homework #1
due Thursday, Feb. 1

1. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}.$$

2. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 2 & 2 & -1 \end{bmatrix}.$$

3. The following matrix has eigenvalues $\lambda = 1, 1, 2, 2$. Is it diagonalisable? Explain.

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 2 & -1 & 1 \\ -1 & 1 & 0 & 2 \\ -2 & 2 & -1 & 3 \end{bmatrix}.$$

4. Suppose A is a diagonalisable matrix and let $k \geq 1$ be an integer. Show that each eigenvector of A is an eigenvector of A^k and conclude that A^k is diagonalisable.

- This assignment is due by Thursday noon, either in class or else in my office.
- Write your name and course (Maths, TP, TSM) on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra II
Homework #2
due Thursday, Feb. 8

1. Let $x_0 = 3$ and $y_0 = 1$. Suppose the sequences x_n, y_n are such that

$$x_n = 3x_{n-1} - 2y_{n-1}, \qquad y_n = 4x_{n-1} + 9y_{n-1}$$

for each $n \geq 1$. Determine each of x_n and y_n explicitly in terms of n.

2. Show that the following matrix is diagonalisable.

$$A = \begin{bmatrix} 7 & 1 & -7 \\ 3 & 3 & -5 \\ 3 & 1 & -3 \end{bmatrix}.$$

3. Find the eigenvalues and the generalised eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

4. Suppose that A is a 4×4 matrix whose first two columns are linearly independent, its third column is equal to the first column and its last column is zero. Find a basis for both the column space and the null space of A. Hint: $A\boldsymbol{e}_3 = A\boldsymbol{e}_1$ and $A\boldsymbol{e}_4 = 0$.

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Linear algebra II
Homework #3
due Thursday, Feb. 15

1. The following matrix has $\lambda = 2$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}.$$

2. The following matrix has $\lambda = 2$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & 6 & 1 \end{bmatrix}.$$

3. Find a Jordan chain of length 2 for the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}.$$

4. Let $\boldsymbol{x} \in \mathbb{R}^3$ be nonzero and let A be the matrix whose columns are $\boldsymbol{x}, 2\boldsymbol{x}, 3\boldsymbol{x}$ in this order. Show that $\boldsymbol{x}$ is an eigenvector of A and find a basis for the null space of A.

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Linear algebra II
Homework #4
due Thursday, Feb. 22

1. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & -2 \\ 4 & 8 & -3 \end{bmatrix}.$$

2. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 4 & -1 \\ 1 & 3 & 1 \end{bmatrix}.$$

3. Suppose A is a 2×2 matrix such that $A^2 = I_2$ and let J be the Jordan form of A. Show that $J^2 = I_2$ and use this fact to conclude that J is diagonal.

4. Suppose A is a 4×4 matrix with characteristic polynomial $f(\lambda) = \lambda^3(\lambda - 1)$ and suppose its column space is two-dimensional. Find the Jordan form of A.

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- Write your name and course (Maths, TP, TSM) on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra II
Homework #5
due Thursday, March 8

1. Let $x_0 = 1$ and $y_0 = 2$. Suppose the sequences x_n, y_n are such that

$$x_n = 8x_{n-1} - 9y_{n-1}, \qquad y_n = x_{n-1} + 2y_{n-1}$$

for each $n \geq 1$. Determine each of x_n and y_n explicitly in terms of n.

2. Which of the following matrices are similar? Explain.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

3. Show that the trace of a square matrix A is the sum of its eigenvalues. Hint: prove the same statement for the Jordan form of A and then use similarity.

4. Let $\boldsymbol{x} \in \mathbb{R}^3$ be nonzero and let A be the matrix whose columns are $\boldsymbol{x}, 2\boldsymbol{x}, 3\boldsymbol{x}$ in this order. Find the Jordan form of A. Hint: the answer depends on the trace of A; show that the null space is two-dimensional and that the eigenvalues are $\lambda = 0, 0, \operatorname{tr} A$.

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- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra II
Homework #6
due Thursday, March 15

1. The following matrix has $\lambda = 1$ as a triple eigenvalue. Use this fact to find its Jordan form, its minimal polynomial and also its inverse.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}.$$

2. The following matrix has eigenvalues $\lambda = 0, 1, 1$. Use this fact to find its Jordan form, its minimal polynomial and also its power A^{2018}.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}.$$

3. Suppose that A is a 2×2 matrix with $\det A = 0$. Use the Cayley-Hamilton theorem to show that $A^2 = (\operatorname{tr} A)A$ and determine A^n explicitly for each integer $n \geq 2$.

4. Let P_2 be the space of all real polynomials of degree at most 2 and let

$$\langle f, g \rangle = \int_0^1 (1 - x) \cdot f(x) g(x)\, dx \quad \text{for all } f, g \in P_2.$$

Find the matrix of this bilinear form with respect to the standard basis.

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Linear algebra II
Homework #7
due Thursday, March 22

1. Consider $\mathbb{R}^3$ with the usual dot product. Use the Gram-Schmidt procedure to find an orthogonal basis, starting with the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + x_1 y_2 + x_2 y_1 + 5 x_2 y_2.$$

Show that this is an inner product and use the Gram-Schmidt procedure to find an orthogonal basis for it, starting with the standard basis of $\mathbb{R}^2$.

3. Define a bilinear form on the space M_{22} of all 2×2 real matrices by setting

$$\langle A, B \rangle = \mathrm{tr}(A^t B)$$

for all 2×2 real matrices A, B. Express this equation in terms of the entries of the two matrices. Is the bilinear form symmetric? Is it positive definite?

4. Find two eigenvectors of A which form an orthonormal basis of $\mathbb{R}^2$ when

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \qquad a, b \neq 0.$$

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Linear algebra II
Homework #8
due Thursday, March 29

1. Find an orthogonal matrix B such that $B^t AB$ is diagonal when

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

2. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form an orthonormal basis of $\mathbb{R}^n$ and consider the $n \times n$ matrix $A = \boldsymbol{v}_2 \boldsymbol{v}_1^t$. Show that $A^2 = 0$ and find the Jordan form of A.

3. Find an orthonormal basis of $\mathbb{R}^3$ that consists entirely of eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

4. Find an orthogonal 3×3 matrix whose first two columns are

$$\boldsymbol{v}_1 = \begin{bmatrix} \cos x \cdot \cos y \\ \sin x \\ \cos x \cdot \sin y \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -\sin y \\ 0 \\ \cos y \end{bmatrix}, \qquad x, y \in \mathbb{R}.$$

- This assignment is due by Thursday noon, either in class or else in my office.
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