Linear algebra II
Homework #1
due Thursday, Feb. 1

1. Find the eigenvalues and the eigenvectors of the matrix

\[ A = \begin{bmatrix} 3 & 2 \\ -1 & 6 \end{bmatrix}. \]

2. Find the eigenvalues and the eigenvectors of the matrix

\[ A = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 3 & -2 \\ 2 & 2 & -1 \end{bmatrix}. \]

3. The following matrix has eigenvalues \( \lambda = 1, 1, 2, 2 \). Is it diagonalisable? Explain.

\[ A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 2 & -1 & 1 \\ -1 & 1 & 0 & 2 \\ -2 & 2 & -1 & 3 \end{bmatrix}. \]

4. Suppose \( A \) is a diagonalisable matrix and let \( k \geq 1 \) be an integer. Show that each
eigenvector of \( A \) is an eigenvector of \( A^k \) and conclude that \( A^k \) is diagonalisable.

- This assignment is due by Thursday noon, either in class or else in my office.
- Write your name and course (Maths, TP, TSM) on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.
Linear algebra II  
Homework #2  
due Thursday, Feb. 8

1. Let \( x_0 = 3 \) and \( y_0 = 1 \). Suppose the sequences \( x_n, y_n \) are such that

\[
x_n = 3x_{n-1} - 2y_{n-1}, \quad y_n = 4x_{n-1} + 9y_{n-1}
\]

for each \( n \geq 1 \). Determine each of \( x_n \) and \( y_n \) explicitly in terms of \( n \).

2. Show that the following matrix is diagonalisable.

\[
A = \begin{bmatrix}
7 & 1 & -7 \\
3 & 3 & -5 \\
3 & 1 & -3
\end{bmatrix}
\]

3. Find the eigenvalues and the generalised eigenvectors of the matrix

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
1 & 3 & -1 \\
0 & 1 & 2
\end{bmatrix}
\]

4. Suppose that \( A \) is a \( 4 \times 4 \) matrix whose first two columns are linearly independent, its third column is equal to the first column and its last column is zero. Find a basis for both the column space and the null space of \( A \). Hint: \( A e_3 = A e_1 \) and \( A e_4 = 0 \).

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1. The following matrix has $\lambda = 2$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}.$$ 

2. The following matrix has $\lambda = 2$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & 6 & 1 \end{bmatrix}.$$ 

3. Find a Jordan chain of length 2 for the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & 5 \end{bmatrix}.$$ 

4. Let $\mathbf{x} \in \mathbb{R}^3$ be nonzero and let $A$ be the matrix whose columns are $\mathbf{x}, 2\mathbf{x}, 3\mathbf{x}$ in this order. Show that $\mathbf{x}$ is an eigenvector of $A$ and find a basis for the null space of $A$. 

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Linear algebra II  
Homework #4  
due Thursday, Feb. 22

1. Find the Jordan form and a Jordan basis for the matrix  
\[ A = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & -2 \\ 4 & 8 & -3 \end{bmatrix}. \]

2. Find the Jordan form and a Jordan basis for the matrix  
\[ A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 4 & -1 \\ 1 & 3 & 1 \end{bmatrix}. \]

3. Suppose \( A \) is a \( 2 \times 2 \) matrix such that \( A^2 = I_2 \) and let \( J \) be the Jordan form of \( A \). Show that \( J^2 = I_2 \) and use this fact to conclude that \( J \) is diagonal.

4. Suppose \( A \) is a \( 4 \times 4 \) matrix with characteristic polynomial \( f(\lambda) = \lambda^3(\lambda - 1) \) and suppose its column space is two-dimensional. Find the Jordan form of \( A \).

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- NO LATE HOMEWORK WILL BE ACCEPTED.
1. Let $x_0 = 1$ and $y_0 = 2$. Suppose the sequences $x_n, y_n$ are such that

$$x_n = 8x_{n-1} - 9y_{n-1}, \quad y_n = x_{n-1} + 2y_{n-1}$$

for each $n \geq 1$. Determine each of $x_n$ and $y_n$ explicitly in terms of $n$.

2. Which of the following matrices are similar? Explain.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}.$$ 

3. Show that the trace of a square matrix $A$ is the sum of its eigenvalues. Hint: prove the same statement for the Jordan form of $A$ and then use similarity.

4. Let $x \in \mathbb{R}^3$ be nonzero and let $A$ be the matrix whose columns are $x, 2x, 3x$ in this order. Find the Jordan form of $A$. Hint: the answer depends on the trace of $A$; show that the null space is two-dimensional and that the eigenvalues are $\lambda = 0, 0, \text{tr } A$.

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1. The following matrix has $\lambda = 1$ as a triple eigenvalue. Use this fact to find its Jordan form, its minimal polynomial and also its inverse.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}.$$ 

2. The following matrix has eigenvalues $\lambda = 0, 1, 1$. Use this fact to find its Jordan form, its minimal polynomial and also its power $A^{2018}$.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}.$$ 

3. Suppose that $A$ is a $2 \times 2$ matrix with $\det A = 0$. Use the Cayley-Hamilton theorem to show that $A^2 = (\tr A)A$ and determine $A^n$ explicitly for each integer $n \geq 2$.

4. Let $P_2$ be the space of all real polynomials of degree at most 2 and let

$$\langle f, g \rangle = \int_0^1 (1 - x) \cdot f(x)g(x) \, dx \quad \text{for all } f, g \in P_2.$$ 

Find the matrix of this bilinear form with respect to the standard basis.

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1. Consider $\mathbb{R}^3$ with the usual dot product. Use the Gram-Schmidt procedure to find an orthogonal basis, starting with the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

2. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle x, y \rangle = x_1y_1 + x_1y_2 + x_2y_1 + 5x_2y_2.$$

Show that this is an inner product and use the Gram-Schmidt procedure to find an orthogonal basis for it, starting with the standard basis of $\mathbb{R}^2$.

3. Define a bilinear form on the space $M_{22}$ of all $2 \times 2$ real matrices by setting

$$\langle A, B \rangle = \text{tr}(A^tB)$$

for all $2 \times 2$ real matrices $A, B$. Express this equation in terms of the entries of the two matrices. Is the bilinear form symmetric? Is it positive definite?

4. Find two eigenvectors of $A$ which form an orthonormal basis of $\mathbb{R}^2$ when

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad a, b \neq 0.$$

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1. Find an orthogonal matrix $B$ such that $B^t A B$ is diagonal when

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}. $$

2. Suppose that $v_1, v_2, \ldots, v_n$ form an orthonormal basis of $\mathbb{R}^n$ and consider the $n \times n$ matrix $A = v_2 v_1^t$. Show that $A^2 = 0$ and find the Jordan form of $A$.

3. Find an orthonormal basis of $\mathbb{R}^3$ that consists entirely of eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}. $$

4. Find an orthogonal $3 \times 3$ matrix whose first two columns are

$$v_1 = \begin{bmatrix} \cos x \cdot \cos y \\ \sin x \\ \cos x \cdot \sin y \end{bmatrix}, \quad v_2 = \begin{bmatrix} -\sin y \\ 0 \\ \cos y \end{bmatrix}, \quad x, y \in \mathbb{R}. $$

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