Linear algebra II
Tutorial problems #1

1. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}.$$

2. Is the following matrix diagonalisable? Why or why not?

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}.$$

3. Find a matrix A that has $\boldsymbol{v}_1$ as an eigenvector with eigenvalue $\lambda_1 = 2$ and $\boldsymbol{v}_2$ as an eigenvector with eigenvalue $\lambda_2 = 5$ when

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

4. Two square matrices A, C are said to be similar, if $C = B^{-1}AB$ for some invertible matrix B. Show that similar matrices have the same characteristic polynomial and also the same eigenvalues. Hint: one has $C - \lambda I = B^{-1}(A - \lambda I)B$.

Linear algebra II
Tutorial problems #2

1. Find a basis for both the null space and the column space of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 2 & 0 & 2 & 6 \\ 1 & 1 & 3 & 4 \end{bmatrix}.$$

2. Find the eigenvalues and the generalised eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -6 & 3 \\ 0 & -1 & 4 \\ 1 & -2 & 2 \end{bmatrix}.$$

3. Show that $\mathcal{W}$ is an A-invariant subspace of $\mathbb{R}^3$ in the case that

$$\mathcal{W} = \operatorname{Span}\left\{ \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \right\}, \qquad A = \begin{bmatrix} 4 & -1 & -5 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}.$$

4. Let $\lambda \in \mathbb{R}$ be a given number. Find all real 2×2 matrices A such that

$$\mathcal{N}(A - \lambda I) = \operatorname{Span}\{\boldsymbol{e}_2\}, \qquad \mathcal{N}(A - \lambda I)^2 = \operatorname{Span}\{\boldsymbol{e}_1, \boldsymbol{e}_2\} = \mathbb{R}^2.$$

Linear algebra II
Tutorial problems #3

1. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ -7 & 5 & 3 \\ -5 & 1 & 6 \end{bmatrix}.$$

2. Suppose that A is a 4×4 matrix whose column space is equal to its null space. Show that A^2 must be the zero matrix and find the Jordan form of A.

3. Suppose that A is a 4×4 matrix whose only eigenvalue is $\lambda = 1$. Suppose also that the column space of $A - I$ is one-dimensional. Find the Jordan form of A.

4. Suppose that A is a 2×2 matrix with $A^2 = I$. Show that A is diagonalisable.

Linear algebra II
Tutorial problems #4

1. The following matrix has $\lambda = 1$ as a triple eigenvalue. Use this fact to find its Jordan form, its minimal polynomial and also its inverse.

$$A = \begin{bmatrix} -1 & 4 & -4 \\ -2 & 5 & -4 \\ -1 & 2 & -1 \end{bmatrix}.$$

2. Suppose A is a non-diagonalisable 3×3 matrix such that $A^3 = A^2$. What are the two possible minimal polynomials of A? What are the two possible Jordan forms?

3. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 3x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2.$$

Find the matrix A of this form with respect to the standard basis and then find the matrix with respect to a basis consisting of eigenvectors of A.

4. Let P_1 be the space of all real polynomials of degree at most 1 and define

$$\langle f, g \rangle = \int_0^1 (x+1)f(x)g(x)\,dx \quad \text{for all } f, g \in P_1.$$

Find the matrix A of this bilinear form with respect to the standard basis.

Linear algebra II
Tutorial problems #5

1. Find an orthogonal matrix B such that $B^t AB$ is diagonal when

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}.$$

2. A real matrix A is called skew-symmetric, if $A^t = -A$. Show that the eigenvalues of such a matrix are purely imaginary, namely of the form $\lambda = iy$ with $y \in \mathbb{R}$.

3. Show that $\boldsymbol{x}^t A\boldsymbol{x} = \lambda||\boldsymbol{x}||^2$ whenever $\boldsymbol{x}$ is a real eigenvector of A with eigenvalue λ. Can a positive definite matrix have a negative eigenvalue? Explain.

4. Find a 2×2 symmetric matrix A with eigenvalues $\lambda = 1, 2$ such that $\boldsymbol{v}_1 = \left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right]$ is an eigenvector of A with eigenvalue $\lambda_1 = 1$.

5. Find an orthonormal basis of $\mathbb{R}^3$ that consists entirely of eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Linear algebra II
Tutorial problems #6

1. Find the values of a for which the matrix A is positive definite when

$$A = \begin{bmatrix} a & 1 & a \\ 1 & a & 1 \\ a & 1 & 2 \end{bmatrix}.$$

2. Prove the converse of the spectral theorem: if A is a real matrix such that $B^t AB$ is diagonal for some orthogonal matrix B, then A must actually be symmetric.

3. Find all $n \times n$ real symmetric matrices A such that $A^3 = I_n$.

4. Show that $A = I_n + \boldsymbol{v}\boldsymbol{v}^t$ is positive definite symmetric for each vector $\boldsymbol{v} \in \mathbb{R}^n$.

5. Find the values of a for which $Q(x, y, z)$ is positive definite when

$$Q(x, y, z) = x^2 + (a + 2)y^2 + az^2 + 2axy + 2axz + 2yz.$$

6. Suppose that A is a real positive definite symmetric matrix. Show that there exists a real positive definite symmetric matrix Q such that $Q^2 = A$.