Linear algebra II
Homework #8 solutions

1. Find an orthogonal matrix B such that B^tAB is diagonal when

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 9\lambda^2 - 18\lambda = -\lambda(\lambda - 3)(\lambda - 6).$$

This gives $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = 6$, while the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Since the eigenvalues are distinct, the eigenvectors are orthogonal to one another. We may thus divide each of them by its length to obtain an orthogonal matrix B such that

$$B = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \quad \Longrightarrow \quad B^tAB = B^{-1}AB = \begin{bmatrix} 0 & & \\ & 3 & \\ & & 6 \end{bmatrix}.$$

2. Let P_1 be the space of all real polynomials of degree at most 1 and let

$$\langle f, g \rangle = \int_{-1}^{1} 3x \cdot f(x)g(x)\,dx \quad \text{for all } f, g \in P_1.$$

Find the matrix A of this bilinear form with respect to the standard basis and then find an orthogonal matrix B such that B^tAB is diagonal.

By definition, the entries of the matrix A are given by the formula

$$a_{ij} = \langle x^{i-1}, x^{j-1} \rangle = \int_{-1}^{1} 3x^{i+j-1}\,dx.$$

This gives $a_{ij} = 0$ when $i + j$ is even and also $a_{ij} = 6/(i + j)$ when $i + j$ is odd, so

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -2$, while the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since the eigenvalues are distinct, the eigenvectors are orthogonal to one another. We may thus divide each of them by its length to obtain an orthogonal matrix B such that

$$B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad \Longrightarrow \quad B^t AB = B^{-1}AB = \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}.$$

3. Show that every eigenvalue λ of a real orthogonal matrix B has absolute value 1. In other words, show that every eigenvalue λ of B satisfies $\lambda\overline{\lambda} = 1$.

Assuming that $\boldsymbol{v}$ is an eigenvector of B with eigenvalue λ, we get

$$\lambda\overline{\lambda}\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \langle \lambda\boldsymbol{v}, \lambda\boldsymbol{v}\rangle = \langle B\boldsymbol{v}, B\boldsymbol{v}\rangle = \langle B^*B\boldsymbol{v}, \boldsymbol{v}\rangle.$$

Since B is real and orthogonal, one has $B^*B = B^tB = I_n$ and this implies that

$$\lambda\overline{\lambda}\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \langle B^*B\boldsymbol{v}, \boldsymbol{v}\rangle = \langle \boldsymbol{v}, \boldsymbol{v}\rangle \quad \Longrightarrow \quad \lambda\overline{\lambda} = 1.$$

4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form an orthonormal basis of $\mathbb{R}^n$ and consider the $n \times n$ matrix $A = I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t$. Show that A is symmetric, orthogonal and diagonalisable.

To say that A is symmetric is to say that $A^t = A$ and this is true because

$$A^t = I_n^t - 2(\boldsymbol{v}_1\boldsymbol{v}_1^t)^t = I_n - 2\boldsymbol{v}_1^{tt}\boldsymbol{v}_1^t = I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t = A.$$

To say that A is orthogonal is to say that $A^tA = I_n$ and this is true because

$$A^tA = AA = (I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t)(I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t) = I_n - 4\boldsymbol{v}_1\boldsymbol{v}_1^t + 4\boldsymbol{v}_1(\boldsymbol{v}_1^t\boldsymbol{v}_1)\boldsymbol{v}_1^t = I_n.$$

Finally, we show that A is diagonalisable. This follows by the spectral theorem, but it can also be verified directly by showing that each $\boldsymbol{v}_i$ is an eigenvector of A. In fact, one has

$$A\boldsymbol{v}_1 = \boldsymbol{v}_1 - 2\boldsymbol{v}_1(\boldsymbol{v}_1^t\boldsymbol{v}_1) = -\boldsymbol{v}_1,$$
$$A\boldsymbol{v}_k = \boldsymbol{v}_k - 2\boldsymbol{v}_1(\boldsymbol{v}_1^t\boldsymbol{v}_k) = \boldsymbol{v}_k$$

for each $2 \leq k \leq n$. This gives n linearly independent eigenvectors, so A is diagonalisable.