1. Consider $\mathbb{R}^3$ with the usual dot product. Use the Gram-Schmidt procedure to find an orthogonal basis, starting with the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$ 

Using the Gram-Schmidt procedure, we let $w_1 = v_1$ and replace the second vector by

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}.$$ 

As for the third vector $v_3$, this needs to be replaced by

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$ 

2. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle x, y \rangle = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 6x_2 y_2.$$ 

Show that this is an inner product and use the Gram-Schmidt procedure to find an orthogonal basis for it, starting with the standard basis of $\mathbb{R}^2$.

The given form is symmetric because its matrix with respect to the standard basis is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}.$$ 

To show that the form is also positive definite, we complete the square to find that

$$\langle x, x \rangle = x_1^2 + 4x_1 x_2 + 6x_2^2 = (x_1 + 2x_2)^2 + 2x_2^2.$$ 

Finally, one may obtain an orthogonal basis by letting $w_1 = e_1$ and

$$w_2 = e_2 - \frac{\langle e_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = e_2 - \frac{e_2^t A e_1}{e_1^t A e_1} e_1 = e_2 - \frac{a_{21}}{a_{11}} e_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$
3. Let  $A = \begin{bmatrix} 1 & 3 \\ 4 & a \end{bmatrix}$. For which values of $a$ is the form $\langle x, y \rangle = x^t A y$ positive definite?

Letting $y = x$ and completing the square, one finds that

$$\langle x, x \rangle = x_1^2 + 7x_1x_2 + ax_2^2 = (x_1 + 7x_2/2)^2 + (a - 49/4)x_2^2.$$  

It easily follows that the given form is positive definite if and only if $a > 49/4$.

4. Let $P_1$ be the space of all real polynomials of degree at most 1 and let

$$\langle f, g \rangle = \int_0^1 (4 - 5x) \cdot f(x)g(x) \, dx \text{ for all } f, g \in P_1.$$  

Is this bilinear form positive definite? Hint: compute $\langle ax + b, ax + b \rangle$ for all $a, b \in \mathbb{R}$.

Every element of $P_1$ has the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}$ and this gives

$$\langle f, f \rangle = \langle ax + b, ax + b \rangle = \int_0^1 (4 - 5x) \cdot (ax + b)^2 \, dx.$$  

We now expand the quadratic factor and then integrate to get

$$\langle f, f \rangle = \int_0^1 (4 - 5x) \cdot (a^2x^2 + 2abx + b^2) \, dx$$

$$= \int_0^1 \left(4b^2 + (8ab - 5b^2)x + (4a^2 - 10ab)x^2 - 5a^2x^3\right) \, dx$$

$$= 4b^2 + \frac{8ab - 5b^2}{2} + \frac{4a^2 - 10ab}{3} - \frac{5a^2}{4}.$$  

To see that the form is positive definite, it remains to rearrange terms and write

$$\langle f, f \rangle = \frac{a^2 + 8ab + 18b^2}{12} = \frac{(a+4b)^2 + 2b^2}{12}.$$