Linear algebra II
Homework #6 solutions

1. The following matrix has $\lambda = -1$ as a triple eigenvalue. Use this fact to find its Jordan form, its minimal polynomial and also its inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 6 & -4 & 3 \\ 2 & -1 & 0 \end{bmatrix}.$$

To find the Jordan form of A, we use row reduction on the matrix $A - \lambda I$ to get

$$A - \lambda I = A + I = \begin{bmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 2 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives one pivot and two free variables, so there are two Jordan blocks and

$$J = \left[\begin{array}{cc|c} -1 & & \\ 1 & -1 & \\ \hline & & -1 \end{array}\right].$$

The minimal polynomial is $m(\lambda) = (\lambda + 1)^2$. Since A satisfies this polynomial, one has

$$A^2 + 2A + I = 0 \quad \Longrightarrow \quad I = -A(2I + A) \quad \Longrightarrow \quad A^{-1} = -2I - A = \begin{bmatrix} -3 & 1 & -1 \\ -6 & 2 & -3 \\ -2 & 1 & -2 \end{bmatrix}.$$

2. The following matrix has eigenvalues $\lambda = 0, 1, 1$. Use this fact to find its Jordan form, its minimal polynomial and also its power A^{2017}.

$$A = \begin{bmatrix} 4 & -5 & 2 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalue $\lambda = 0$ is simple, so it contributes an 1×1 block. When $\lambda = 1$, we have

$$A - \lambda I = A - I = \begin{bmatrix} 3 & -5 & 2 \\ 2 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and this gives a single 2×2 block with eigenvalue $\lambda = 1$. The Jordan form is thus

$$J = \left[\begin{array}{cc|c} 1 & & \\ 1 & 1 & \\ \hline & & 0 \end{array}\right]$$

and the minimal polynomial is $m(\lambda) = \lambda(\lambda-1)^2$. Since A satisfies this polynomial,

$$\begin{aligned} A^3 - 2A^2 + A = 0 \quad &\implies \quad A^3 = 2A^2 - A \\ &\implies \quad A^4 = 2A^3 - A^2 = 2(2A^2 - A) - A^2 = 3A^2 - 2A \\ &\implies \quad A^5 = 3A^3 - 2A^2 = 3(2A^2 - A) - 2A^2 = 4A^2 - 3A. \end{aligned}$$

It follows by induction that $A^n = (n-1)A^2 - (n-2)A$ for each $n \geq 3$ and this gives

$$A^{2017} = 2016 \begin{bmatrix} 6 & -8 & 3 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} - 2015 \begin{bmatrix} 4 & -5 & 2 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4036 & -6053 & 2018 \\ 4034 & -6050 & 2017 \\ 4032 & -6047 & 2016 \end{bmatrix}.$$

3. Let P_2 be the space of all real polynomials of degree at most 2 and let

$$\langle f, g \rangle = \int_0^1 (3-x) \cdot f(x)g(x)\, dx \quad \text{for all } f, g \in P_2.$$

Find the matrix of this bilinear form with respect to the standard basis.

The standard basis consists of the polynomials $\boldsymbol{v}_1 = 1$, $\boldsymbol{v}_2 = x$, $\boldsymbol{v}_3 = x^2$ and this gives

$$\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \langle x^{i-1}, x^{j-1} \rangle = \int_0^1 (3-x) \cdot x^{i+j-2}\, dx = \frac{3}{i+j-1} - \frac{1}{i+j}$$

for all integers $1 \leq i, j \leq 3$. Since the matrix of the form has entries $a_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$, we get

$$A = \begin{bmatrix} 5/2 & 7/6 & 3/4 \\ 7/6 & 3/4 & 11/20 \\ 3/4 & 11/20 & 13/30 \end{bmatrix}.$$

4. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 4x_1y_1 + 2x_1y_2 + 2x_2y_1 + 7x_2y_2.$$

Find the matrix A of this form with respect to the standard basis and then find the matrix with respect to a basis consisting of eigenvectors of A.

The matrix of the bilinear form $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i,j} a_{ij}x_iy_j$ with respect to the standard basis is the matrix A whose (i,j)th entry is the coefficient of x_iy_j. In our case, we have

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}.$$

Let us now compute the eigenvectors of this matrix. The characteristic polynomial is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 11\lambda + 24 = (\lambda - 3)(\lambda - 8),$$

so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 8$. The corresponding eigenvectors turn out to be

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Finally, we find the matrix M with respect to the basis $\boldsymbol{v}_1, \boldsymbol{v}_2$. Since $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^t A \boldsymbol{y}$, we get

$$m_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \boldsymbol{v}_i^t A \boldsymbol{v}_j = \lambda_j \boldsymbol{v}_i^t \boldsymbol{v}_j = \lambda_j (\boldsymbol{v}_i \cdot \boldsymbol{v}_j) \quad \Longrightarrow \quad M = \begin{bmatrix} 15 & 0 \\ 0 & 40 \end{bmatrix}.$$