1. Find the Jordan form and a Jordan basis for the matrix

\[ A = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -2 & 6 \\ 2 & -1 & 3 \end{bmatrix}. \]

The characteristic polynomial of the given matrix is

\[ f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = (2 - \lambda)(\lambda - 1)^2, \]

so its eigenvalues are \( \lambda = 1, 1, 2 \). The corresponding null spaces are easily found to be

\[ \mathcal{N}(A - I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 2I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}. \]

These contain 3 linearly independent eigenvectors, so \( A \) is diagonalisable and

\[ B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \implies J = B^{-1}AB = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}. \]

2. Find the Jordan form and a Jordan basis for the matrix

\[ A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 4 & -1 \\ 2 & 0 & 2 \end{bmatrix}. \]

The characteristic polynomial of the given matrix is

\[ f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 8\lambda^2 - 20\lambda + 16 = (4 - \lambda)(\lambda - 2)^2, \]

so its eigenvalues are \( \lambda = 4, 2, 2 \). The corresponding null spaces are easily found to be

\[ \mathcal{N}(A - 4I) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 2I) = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}. \]

This implies that \( A \) is not diagonalisable and that its Jordan form is

\[ J = B^{-1}AB = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}. \]
To find a Jordan basis, we need to find vectors $v_1, v_2, v_3$ such that $v_1$ is an eigenvector with eigenvalue $\lambda = 4$ and $v_2, v_3$ is a Jordan chain with eigenvalue $\lambda = 2$. In our case, we have

$$\mathcal{N}(A - 2I)^2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\},$$

so it easily follows that a Jordan basis is provided by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = (A - 2I)v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

3. Suppose that $A$ is a $2 \times 2$ matrix with $A^2 = A$. Show that $A$ is diagonalisable.

Suppose that $A$ is not diagonalisable. Then its eigenvalues are not distinct, so there is a double eigenvalue $\lambda$ and the Jordan form is

$$J = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} \implies J^2 = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda \\ 2\lambda & \lambda^2 \end{bmatrix}.$$

Write $J = B^{-1}AB$ for some invertible matrix $B$. Since $A^2 = A$, we must also have

$$J^2 = B^{-1}AB \cdot B^{-1}AB = B^{-1}A^2B = B^{-1}AB = J = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix}.$$

Comparing the last two equations now gives $\lambda^2 = \lambda$ and $2\lambda = 1$, a contradiction.

4. A $5 \times 5$ matrix $A$ has characteristic polynomial $f(\lambda) = \lambda^4(2 - \lambda)$ and its column space is two-dimensional. Find the dimension of the column space of $A^2$.

The eigenvalue $\lambda = 0$ has multiplicity 4 and the number of Jordan chains is

$$\dim\mathcal{N}(A) = 5 - \dim\mathcal{C}(A) = 3.$$

In particular, the Jordan chain diagram is $\bullet \bullet$ and we have $\dim\mathcal{N}(A^2) = 4$, so

$$\dim\mathcal{C}(A^2) = 5 - \dim\mathcal{N}(A^2) = 1.$$