1. Find a basis for both the null space and the column space of the matrix

\[ A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 3 & 1 & 5 & 0 \\ 4 & 2 & 6 & 2 \\ 4 & 3 & 5 & 5 \end{bmatrix}. \]

The reduced row echelon form of \( A \) is easily found to be

\[ R = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

Since the pivots appear in the first and the second columns, this implies that

\[ \text{C}(A) = \text{Span}\{ Ae_1, Ae_2 \} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}. \]

The null space of \( A \) is the same as the null space of \( R \). It can be expressed in the form

\[ \mathcal{N}(A) = \left\{ \begin{bmatrix} -2x_3 + x_4 \\ x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}. \]

2. Show that the following matrix is diagonalisable.

\[ A = \begin{bmatrix} 7 & 1 & -3 \\ 3 & 5 & -3 \\ 5 & 1 & -1 \end{bmatrix}. \]

The eigenvalues of \( A \) are the roots of the characteristic polynomial

\[ f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 11\lambda^2 - 38\lambda + 40 = -(\lambda - 2)(\lambda - 4)(\lambda - 5). \]

Since the eigenvalues of \( A \) are distinct, we conclude that \( A \) is diagonalisable.
3. Find the eigenvalues and the generalised eigenvectors of the matrix

\[
A = \begin{bmatrix}
-1 & 1 & 2 \\
-3 & 1 & 3 \\
-5 & 1 & 6 \\
\end{bmatrix}.
\]

The eigenvalues of \(A\) are the roots of the characteristic polynomial

\[
f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (4 - \lambda)(\lambda - 1)^2.
\]

When it comes to the eigenvalue \(\lambda = 4\), one can easily check that

\[
\mathcal{N}(A - 4I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}, \quad \mathcal{N}(A - 4I)^2 = \mathcal{N}(A - 4I).
\]

This implies that \(\mathcal{N}(A - 4I)^j = \mathcal{N}(A - 4I)\) for all \(j \geq 1\), so we have found all generalised eigenvectors with \(\lambda = 4\). When it comes to the eigenvalue \(\lambda = 1\), one similarly has

\[
\mathcal{N}(A - I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{N}(A - I)^2 = \mathcal{N}(A - I)^3 = \mathcal{N}(A - I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]

In view of the general theory, we must thus have \(\mathcal{N}(A - I)^j = \mathcal{N}(A - I)^2\) for all \(j \geq 2\).

4. Suppose that \(v_1, v_2, \ldots, v_k\) form a basis for the null space of a square matrix \(A\) and that \(C = B^{-1}AB\) for some invertible matrix \(B\). Find a basis for the null space of \(C\).

We note that a vector \(v\) lies in the null space of \(C = B^{-1}AB\) if and only if

\[
B^{-1}ABv = 0 \iff ABv = 0 \iff Bv \in \mathcal{N}(A) \iff Bv = \sum_{i=1}^{k} c_i v_i
\]

for some scalar coefficients \(c_i\). In particular, the vectors \(B^{-1}v_i\) span the null space of \(C\). To show that these vectors are also linearly independent, we note that

\[
\sum_{i=1}^{k} c_i B^{-1}v_i = 0 \iff \sum_{i=1}^{k} c_i v_i = 0 \iff c_i = 0 \text{ for all } i.
\]