Linear algebra II
Homework #1 solutions

1. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 6 \\ 2 & 5 \end{bmatrix}.$$

Since $\operatorname{tr} A = 9$ and $\det A = 20 - 12 = 8$, the characteristic polynomial is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 9\lambda + 8 = (\lambda - 1)(\lambda - 8).$$

The eigenvectors with eigenvalue $\lambda = 1$ satisfy the system $A\boldsymbol{v} = \boldsymbol{v}$, namely

$$(A - I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad x + 2y = 0 \quad \Longrightarrow \quad x = -2y.$$

This means that every eigenvector with eigenvalue $\lambda = 1$ must have the form

$$\boldsymbol{v} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad y \neq 0.$$

Similarly, the eigenvectors with eigenvalue $\lambda = 8$ are solutions of $A\boldsymbol{v} = 8\boldsymbol{v}$, so

$$(A - 8I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad 2x - 3y = 0 \quad \Longrightarrow \quad x = 3y/2$$

and every eigenvector with eigenvalue $\lambda = 8$ must have the form

$$\boldsymbol{v} = \begin{bmatrix} 3y/2 \\ y \end{bmatrix} = y \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}, \qquad y \neq 0.$$

2. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & -9 & 5 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = -(\lambda - 2)^2(\lambda - 3).$$

The eigenvectors of A are nonzero vectors in the null spaces

$$\mathcal{N}(A - 2I) = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 3I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

3. The following matrix has eigenvalues $\lambda = 1, 1, 1, 4$. Is it diagonalisable? Explain.

$$A = \begin{bmatrix} 3 & 1 & -3 & 3 \\ 2 & 2 & -3 & 3 \\ 2 & 1 & -2 & 3 \\ 3 & 0 & -3 & 4 \end{bmatrix}.$$

When it comes to the eigenvalue $\lambda = 1$, row reduction of $A - \lambda I$ gives

$$A - \lambda I = \begin{bmatrix} 2 & 1 & -3 & 3 \\ 2 & 1 & -3 & 3 \\ 2 & 1 & -3 & 3 \\ 3 & 0 & -3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so there are 2 pivots and 2 linearly independent eigenvectors. When $\lambda = 4$, we similarly get

$$A - \lambda I = \begin{bmatrix} -1 & 1 & -3 & 3 \\ 2 & -2 & -3 & 3 \\ 2 & 1 & -6 & 3 \\ 3 & 0 & -3 & 0 \end{bmatrix} \longrightarrow \quad \cdots \quad \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so there are 3 pivots and 1 linearly independent eigenvector. This gives a total of 3 linearly independent eigenvectors, so A is not diagonalisable. One does not really need to find the eigenvectors in this case, but those are nonzero elements of the null spaces

$$\mathcal{N}(A - I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 4I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

4. Suppose A is a square matrix which is both diagonalisable and invertible. Show that every eigenvector of A is an eigenvector of A^{-1} and that A^{-1} is diagonalisable.

Suppose that $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ. Then λ is nonzero because

$$\lambda = 0 \quad \Longrightarrow \quad A\boldsymbol{v} = \lambda\boldsymbol{v} = 0 \quad \Longrightarrow \quad A^{-1}A\boldsymbol{v} = 0 \quad \Longrightarrow \quad \boldsymbol{v} = 0$$

and eigenvectors are nonzero. To see that $\boldsymbol{v}$ is also an eigenvector of A^{-1}, we note that

$$\lambda\boldsymbol{v} = A\boldsymbol{v} \quad \Longrightarrow \quad \lambda A^{-1}\boldsymbol{v} = \boldsymbol{v} \quad \Longrightarrow \quad A^{-1}\boldsymbol{v} = \lambda^{-1}\boldsymbol{v}.$$

This proves the first part. To prove the second, assume that A is an $n \times n$ matrix. It must then have n linearly independent eigenvectors which form a matrix B such that $B^{-1}AB$ is diagonal. These vectors are also eigenvectors of A^{-1}, so $B^{-1}A^{-1}B$ is diagonal as well.