Linear algebra II
Homework #1 solutions

1. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 6 \\ 2 & 5 \end{bmatrix}.$$

Since $\operatorname{tr} A = 9$ and $\det A = 20 - 12 = 8$, the characteristic polynomial is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 9\lambda + 8 = (\lambda - 1)(\lambda - 8).$$

The eigenvectors with eigenvalue $\lambda = 1$ satisfy the system $A\boldsymbol{v} = \boldsymbol{v}$, namely

$$(A - I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad x + 2y = 0 \quad \Longrightarrow \quad x = -2y.$$

This means that every eigenvector with eigenvalue $\lambda = 1$ must have the form

$$\boldsymbol{v} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad y \neq 0.$$

Similarly, the eigenvectors with eigenvalue $\lambda = 8$ are solutions of $A\boldsymbol{v} = 8\boldsymbol{v}$, so

$$(A - 8I)\boldsymbol{v} = 0 \quad \Longrightarrow \quad \begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad 2x - 3y = 0 \quad \Longrightarrow \quad x = 3y/2$$

and every eigenvector with eigenvalue $\lambda = 8$ must have the form

$$\boldsymbol{v} = \begin{bmatrix} 3y/2 \\ y \end{bmatrix} = y \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}, \qquad y \neq 0.$$

2. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & -9 & 5 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 7\lambda^2 - 16\lambda + 12 = -(\lambda - 2)^2(\lambda - 3).$$

The eigenvectors of A are nonzero vectors in the null spaces

$$\mathcal{N}(A - 2I) = \operatorname{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 3I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

3. The following matrix has eigenvalues $\lambda = 1, 1, 1, 4$. Is it diagonalisable? Explain.

$$A = \begin{bmatrix} 3 & 1 & -3 & 3 \\ 2 & 2 & -3 & 3 \\ 2 & 1 & -2 & 3 \\ 3 & 0 & -3 & 4 \end{bmatrix}.$$

When it comes to the eigenvalue $\lambda = 1$, row reduction of $A - \lambda I$ gives

$$A - \lambda I = \begin{bmatrix} 2 & 1 & -3 & 3 \\ 2 & 1 & -3 & 3 \\ 2 & 1 & -3 & 3 \\ 3 & 0 & -3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so there are 2 pivots and 2 linearly independent eigenvectors. When $\lambda = 4$, we similarly get

$$A - \lambda I = \begin{bmatrix} -1 & 1 & -3 & 3 \\ 2 & -2 & -3 & 3 \\ 2 & 1 & -6 & 3 \\ 3 & 0 & -3 & 0 \end{bmatrix} \longrightarrow \quad \cdots \quad \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so there are 3 pivots and 1 linearly independent eigenvector. This gives a total of 3 linearly independent eigenvectors, so A is not diagonalisable. One does not really need to find the eigenvectors in this case, but those are nonzero elements of the null spaces

$$\mathcal{N}(A - I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 4I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

4. Suppose A is a square matrix which is both diagonalisable and invertible. Show that every eigenvector of A is an eigenvector of A^{-1} and that A^{-1} is diagonalisable.

Suppose that $\boldsymbol{v}$ is an eigenvector of A with eigenvalue λ. Then λ is nonzero because

$$\lambda = 0 \quad \Longrightarrow \quad A\boldsymbol{v} = \lambda\boldsymbol{v} = 0 \quad \Longrightarrow \quad A^{-1}A\boldsymbol{v} = 0 \quad \Longrightarrow \quad \boldsymbol{v} = 0$$

and eigenvectors are nonzero. To see that $\boldsymbol{v}$ is also an eigenvector of A^{-1}, we note that

$$\lambda\boldsymbol{v} = A\boldsymbol{v} \quad \Longrightarrow \quad \lambda A^{-1}\boldsymbol{v} = \boldsymbol{v} \quad \Longrightarrow \quad A^{-1}\boldsymbol{v} = \lambda^{-1}\boldsymbol{v}.$$

This proves the first part. To prove the second, assume that A is an $n \times n$ matrix. It must then have n linearly independent eigenvectors which form a matrix B such that $B^{-1}AB$ is diagonal. These vectors are also eigenvectors of A^{-1}, so $B^{-1}A^{-1}B$ is diagonal as well.

Linear algebra II
Homework #2 solutions

1. Find a basis for both the null space and the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 3 & 1 & 5 & 0 \\ 4 & 2 & 6 & 2 \\ 4 & 3 & 5 & 5 \end{bmatrix}.$$

The reduced row echelon form of A is easily found to be

$$R = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the pivots appear in the first and the second columns, this implies that

$$\mathcal{C}(A) = \operatorname{Span}\{A\boldsymbol{e}_1, A\boldsymbol{e}_2\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \right\}.$$

The null space of A is the same as the null space of R. It can be expressed in the form

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -2x_3 + x_4 \\ x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} : x_3, x_4 \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2. Show that the following matrix is diagonalisable.

$$A = \begin{bmatrix} 7 & 1 & -3 \\ 3 & 5 & -3 \\ 5 & 1 & -1 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 11\lambda^2 - 38\lambda + 40 = -(\lambda - 2)(\lambda - 4)(\lambda - 5).$$

Since the eigenvalues of A are distinct, we conclude that A is diagonalisable.

3. Find the eigenvalues and the generalised eigenvectors of the matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ -3 & 1 & 3 \\ -5 & 1 & 6 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (4 - \lambda)(\lambda - 1)^2.$$

When it comes to the eigenvalue $\lambda = 4$, one can easily check that

$$\mathcal{N}(A - 4I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 4I)^2 = \mathcal{N}(A - 4I).$$

This implies that $\mathcal{N}(A - 4I)^j = \mathcal{N}(A - 4I)$ for all $j \geq 1$, so we have found all generalised eigenvectors with $\lambda = 4$. When it comes to the eigenvalue $\lambda = 1$, one similarly has

$$\mathcal{N}(A - I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - I)^2 = \mathcal{N}(A - I)^3 = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

In view of the general theory, we must thus have $\mathcal{N}(A - I)^j = \mathcal{N}(A - I)^2$ for all $j \geq 2$.

4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ form a basis for the null space of a square matrix A and that $C = B^{-1}AB$ for some invertible matrix B. Find a basis for the null space of C.

We note that a vector $\boldsymbol{v}$ lies in the null space of $C = B^{-1}AB$ if and only if

$$B^{-1}AB\boldsymbol{v} = 0 \quad\Longleftrightarrow\quad AB\boldsymbol{v} = 0 \quad\Longleftrightarrow\quad B\boldsymbol{v} \in \mathcal{N}(A) \quad\Longleftrightarrow\quad B\boldsymbol{v} = \sum_{i=1}^{k} c_i \boldsymbol{v}_i$$

for some scalar coefficients c_i. In particular, the vectors $B^{-1}\boldsymbol{v}_i$ span the null space of C. To show that these vectors are also linearly independent, we note that

$$\sum_{i=1}^{k} c_i B^{-1}\boldsymbol{v}_i = 0 \quad\Longleftrightarrow\quad \sum_{i=1}^{k} c_i \boldsymbol{v}_i = 0 \quad\Longleftrightarrow\quad c_i = 0 \text{ for all } i.$$

Linear algebra II
Homework #3 solutions

1. The following matrix has $\lambda = 4$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 1 & 5 & 1 \\ -2 & 7 & 1 \\ -1 & 2 & 4 \end{bmatrix}.$$

In this case, the null space of $A - 4I$ is one-dimensional, as row reduction gives

$$A - 4I = \begin{bmatrix} -3 & 5 & 1 \\ -2 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Similarly, the null space of $(A - 4I)^2$ is two-dimensional because

$$(A - 4I)^2 = \begin{bmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

while $(A - 4I)^3$ is the zero matrix, so its null space is three-dimensional. The diagram of Jordan chains is then $\begin{smallmatrix}\bullet\\\bullet\\\bullet\end{smallmatrix}$ and there is a single 3×3 Jordan block, namely

$$J = \begin{bmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{bmatrix} = \begin{bmatrix} 4 & & \\ 1 & 4 & \\ & 1 & 4 \end{bmatrix}.$$

2. The following matrix has $\lambda = 4$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 6 & 2 \\ -1 & 1 & 5 \end{bmatrix}.$$

In this case, the null space of $A - 4I$ is two-dimensional, as row reduction gives

$$A - 4I = \begin{bmatrix} -3 & 3 & 3 \\ -2 & 2 & 2 \\ -1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, $(A - 4I)^2$ is the zero matrix, so its null space is three-dimensional. Thus, the diagram of Jordan chains is $\begin{smallmatrix}\bullet&\bullet\\\bullet&\end{smallmatrix}$ and there is a Jordan chain of length 2 as well as a Jordan chain of length 1. These Jordan chains give a 2×2 block and an 1×1 block, so

$$J = \left[\begin{array}{cc|c} 4 & & \\ 1 & 4 & \\ \hline & & 4 \end{array}\right].$$

3. Find a Jordan chain of length 2 for the matrix

$$A = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}.$$

The eigenvalues of the given matrix are the roots of the characteristic polynomial

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,$$

so $\lambda = 3$ is the only eigenvalue. Using row reduction, we now get

$$A - 3I = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

so the null space of $A - 3I$ is one-dimensional. On the other hand, $(A - 3I)^2$ is the zero matrix, so its null space is two-dimensional. To find a Jordan chain of length 2, we pick a vector $\boldsymbol{v}_1$ that lies in the latter null space, but not in the former. We can always take

$$\boldsymbol{v}_1 = \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{v}_2 = (A - 3I)\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

but there are obviously infinitely many choices. Another possible choice would be

$$\boldsymbol{v}_1 = \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{v}_2 = (A - 3I)\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

4. Let A be a 3×3 matrix that has $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ as a Jordan chain of length 3 and let B be the matrix whose columns are $\boldsymbol{v}_3, \boldsymbol{v}_2, \boldsymbol{v}_1$ (in the order listed). Compute $B^{-1}AB$.

Suppose the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a Jordan chain with eigenvalue λ, in which case

$$(A - \lambda I)\boldsymbol{v}_1 = \boldsymbol{v}_2, \qquad (A - \lambda I)\boldsymbol{v}_2 = \boldsymbol{v}_3, \qquad (A - \lambda I)\boldsymbol{v}_3 = 0.$$

To find the columns of $B^{-1}AB$, we need to find the coefficients that one needs in order to express the vectors $A\boldsymbol{v}_3, A\boldsymbol{v}_2, A\boldsymbol{v}_1$ in terms of the given basis. By above, we have

$$A\boldsymbol{v}_3 = \lambda \boldsymbol{v}_3, \qquad A\boldsymbol{v}_2 = \boldsymbol{v}_3 + \lambda \boldsymbol{v}_2, \qquad A\boldsymbol{v}_1 = \boldsymbol{v}_2 + \lambda \boldsymbol{v}_1.$$

Reading the coefficients of the vectors $\boldsymbol{v}_3, \boldsymbol{v}_2, \boldsymbol{v}_1$ in the given order, we conclude that

$$B^{-1}AB = \begin{bmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}.$$

Linear algebra II
Homework #4 solutions

1. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -2 & 6 \\ 2 & -1 & 3 \end{bmatrix}.$$

The characteristic polynomial of the given matrix is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = (2-\lambda)(\lambda-1)^2,$$

so its eigenvalues are $\lambda = 1, 1, 2$. The corresponding null spaces are easily found to be

$$\mathcal{N}(A - I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 2I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

These contain 3 linearly independent eigenvectors, so A is diagonalisable and

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix} \quad \Longrightarrow \quad J = B^{-1}AB = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}.$$

2. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 4 & -1 \\ 2 & 0 & 2 \end{bmatrix}.$$

The characteristic polynomial of the given matrix is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 8\lambda^2 - 20\lambda + 16 = (4-\lambda)(\lambda-2)^2,$$

so its eigenvalues are $\lambda = 4, 2, 2$. The corresponding null spaces are easily found to be

$$\mathcal{N}(A - 4I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 2I) = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

This implies that A is not diagonalisable and that its Jordan form is

$$J = B^{-1}AB = \left[\begin{array}{c|cc} 4 & & \\ \hline & 2 & \\ & 1 & 2 \end{array}\right].$$

To find a Jordan basis, we need to find vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ such that $\boldsymbol{v}_1$ is an eigenvector with eigenvalue $\lambda = 4$ and $\boldsymbol{v}_2, \boldsymbol{v}_3$ is a Jordan chain with eigenvalue $\lambda = 2$. In our case, we have

$$\mathcal{N}(A-2I)^2 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\},$$

so it easily follows that a Jordan basis is provided by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_3 = (A-2I)\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

3. Suppose that A is a 2×2 matrix with $A^2 = A$. Show that A is diagonalisable.

Suppose that A is not diagonalisable. Then its eigenvalues are not distinct, so there is a double eigenvalue λ and the Jordan form is

$$J = \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix} \quad \Longrightarrow \quad J^2 = \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & \\ 2\lambda & \lambda^2 \end{bmatrix}.$$

Write $J = B^{-1}AB$ for some invertible matrix B. Since $A^2 = A$, we must also have

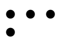
$$J^2 = B^{-1}AB \cdot B^{-1}AB = B^{-1}A^2B = B^{-1}AB = J = \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix}.$$

Comparing the last two equations now gives $\lambda^2 = \lambda$ and $2\lambda = 1$, a contradiction.

4. A 5×5 matrix A has characteristic polynomial $f(\lambda) = \lambda^4(2-\lambda)$ and its column space is two-dimensional. Find the dimension of the column space of A^2.

The eigenvalue $\lambda = 0$ has multiplicity 4 and the number of Jordan chains is

$$\dim \mathcal{N}(A) = 5 - \dim \mathcal{C}(A) = 3.$$

In particular, the Jordan chain diagram is  and we have $\dim \mathcal{N}(A^2) = 4$, so

$$\dim \mathcal{C}(A^2) = 5 - \dim \mathcal{N}(A^2) = 1.$$

Linear algebra II
Homework #5 solutions

1. Let $x_0 = 1$ and $y_0 = -2$. Suppose the sequences x_n, y_n are such that

$$x_n = 4x_{n-1} - y_{n-1}, \qquad y_n = x_{n-1} + 2y_{n-1}$$

for each $n \geq 1$. Determine each of x_n and y_n explicitly in terms of n.

One can express this problem in terms of matrices by writing

$$\boldsymbol{u}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = A\boldsymbol{u}_{n-1} \quad \Longrightarrow \quad \boldsymbol{u}_n = A^n \boldsymbol{u}_0.$$

Let us now focus on computing A^n. The characteristic polynomial of A is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,$$

so the only eigenvalue is $\lambda = 3$. As one can easily check, the corresponding null space is

$$\mathcal{N}(A - 3I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\},$$

while $(A - 3I)^2$ is the zero matrix. Thus, a Jordan basis is provided by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = (A - 3I)\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Letting B be the matrix whose columns are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, we must thus have

$$J = B^{-1}AB = \begin{bmatrix} 3 & \\ 1 & 3 \end{bmatrix} \quad \Longrightarrow \quad J^n = B^{-1}A^nB = \begin{bmatrix} 3^n & \\ n3^{n-1} & 3^n \end{bmatrix}.$$

Once we now solve this equation for A^n, we find that

$$A^n = BJ^nB^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3^n & \\ n3^{n-1} & 3^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (n+3)3^{n-1} & -n3^{n-1} \\ n3^{n-1} & (3-n)3^{n-1} \end{bmatrix}.$$

In particular, the sequences x_n, y_n are given explicitly by

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \boldsymbol{u}_n = A^n \boldsymbol{u}_0 = \begin{bmatrix} (n+1)3^n \\ (n-2)3^n \end{bmatrix}.$$

2. Which of the following matrices are similar? Explain.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Two matrices are similar if and only if they have the same Jordan form. In this case, A is already in Jordan form, all matrices have $\lambda = 2$ as their only eigenvalue, while

$$\dim \mathcal{N}(B - 2I) = 1, \qquad \dim \mathcal{N}(C - 2I) = 2.$$

This means that the Jordan form of B consists of one block, so B is similar to A. On the other hand, the Jordan form of C contains two blocks, so C is not similar to either A or B.

3. Let J be a 4×4 Jordan block with eigenvalue $\lambda = 0$. Find the Jordan form of J^2.

Since J is a Jordan block with eigenvalue $\lambda = 0$, it is easy to check that

$$J = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & 0 \end{bmatrix} \implies J^2 = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 1 & 0 & 0 & \\ & 1 & 0 & 0 \end{bmatrix} \implies J^4 = 0.$$

In particular, the null space of $A = J^2$ is two-dimensional and the null space of $A^2 = J^4$ is four-dimensional, so the Jordan chain diagram is $\begin{smallmatrix} \bullet & \bullet \\ \bullet & \bullet \end{smallmatrix}$ and the Jordan form is

$$J' = \left[\begin{array}{cc|cc} 0 & & & \\ 1 & 0 & & \\ \hline & & 0 & \\ & & 1 & 0 \end{array}\right].$$

4. Use the Cayley-Hamilton theorem to compute A^{2017} in the case that

$$A = \begin{bmatrix} 3 & -2 & -3 \\ 2 & -2 & -2 \\ 3 & -2 & -3 \end{bmatrix}.$$

The characteristic polynomial of the given matrix is $f(\lambda) = -\lambda^3 - 2\lambda^2$, so one has

$$\begin{aligned} -A^3 - 2A^2 = 0 \quad &\implies \quad A^3 = -2A^2 \\ &\implies \quad A^4 = -2A^3 = (-2)^2 A^2. \end{aligned}$$

It follows by induction that $A^n = (-2)^{n-2} A^2$ for each $n \geq 3$ and this finally gives

$$A^{2017} = (-2)^{2015} A^2 = -2^{2015} \begin{bmatrix} -4 & 4 & 4 \\ -4 & 4 & 4 \\ -4 & 4 & 4 \end{bmatrix} = 2^{2017} \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Linear algebra II
Homework #6 solutions

1. The following matrix has $\lambda = -1$ as a triple eigenvalue. Use this fact to find its Jordan form, its minimal polynomial and also its inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 6 & -4 & 3 \\ 2 & -1 & 0 \end{bmatrix}.$$

To find the Jordan form of A, we use row reduction on the matrix $A - \lambda I$ to get

$$A - \lambda I = A + I = \begin{bmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 2 & -1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives one pivot and two free variables, so there are two Jordan blocks and

$$J = \left[\begin{array}{cc|c} -1 & & \\ 1 & -1 & \\ \hline & & -1 \end{array}\right].$$

The minimal polynomial is $m(\lambda) = (\lambda + 1)^2$. Since A satisfies this polynomial, one has

$$A^2 + 2A + I = 0 \quad \Longrightarrow \quad I = -A(2I + A) \quad \Longrightarrow \quad A^{-1} = -2I - A = \begin{bmatrix} -3 & 1 & -1 \\ -6 & 2 & -3 \\ -2 & 1 & -2 \end{bmatrix}.$$

2. The following matrix has eigenvalues $\lambda = 0, 1, 1$. Use this fact to find its Jordan form, its minimal polynomial and also its power A^{2017}.

$$A = \begin{bmatrix} 4 & -5 & 2 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The eigenvalue $\lambda = 0$ is simple, so it contributes an 1×1 block. When $\lambda = 1$, we have

$$A - \lambda I = A - I = \begin{bmatrix} 3 & -5 & 2 \\ 2 & -3 & 1 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and this gives a single 2×2 block with eigenvalue $\lambda = 1$. The Jordan form is thus

$$J = \left[\begin{array}{cc|c} 1 & & \\ 1 & 1 & \\ \hline & & 0 \end{array}\right]$$

and the minimal polynomial is $m(\lambda) = \lambda(\lambda-1)^2$. Since A satisfies this polynomial,

$$\begin{aligned} A^3 - 2A^2 + A = 0 \quad &\implies \quad A^3 = 2A^2 - A \\ &\implies \quad A^4 = 2A^3 - A^2 = 2(2A^2 - A) - A^2 = 3A^2 - 2A \\ &\implies \quad A^5 = 3A^3 - 2A^2 = 3(2A^2 - A) - 2A^2 = 4A^2 - 3A. \end{aligned}$$

It follows by induction that $A^n = (n-1)A^2 - (n-2)A$ for each $n \geq 3$ and this gives

$$A^{2017} = 2016 \begin{bmatrix} 6 & -8 & 3 \\ 4 & -5 & 2 \\ 2 & -2 & 1 \end{bmatrix} - 2015 \begin{bmatrix} 4 & -5 & 2 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 4036 & -6053 & 2018 \\ 4034 & -6050 & 2017 \\ 4032 & -6047 & 2016 \end{bmatrix}.$$

3. Let P_2 be the space of all real polynomials of degree at most 2 and let

$$\langle f, g \rangle = \int_0^1 (3-x) \cdot f(x)g(x)\, dx \quad \text{for all } f, g \in P_2.$$

Find the matrix of this bilinear form with respect to the standard basis.

The standard basis consists of the polynomials $\boldsymbol{v}_1 = 1$, $\boldsymbol{v}_2 = x$, $\boldsymbol{v}_3 = x^2$ and this gives

$$\langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \langle x^{i-1}, x^{j-1} \rangle = \int_0^1 (3-x) \cdot x^{i+j-2}\, dx = \frac{3}{i+j-1} - \frac{1}{i+j}$$

for all integers $1 \leq i, j \leq 3$. Since the matrix of the form has entries $a_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$, we get

$$A = \begin{bmatrix} 5/2 & 7/6 & 3/4 \\ 7/6 & 3/4 & 11/20 \\ 3/4 & 11/20 & 13/30 \end{bmatrix}.$$

4. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 4x_1y_1 + 2x_1y_2 + 2x_2y_1 + 7x_2y_2.$$

Find the matrix A of this form with respect to the standard basis and then find the matrix with respect to a basis consisting of eigenvectors of A.

The matrix of the bilinear form $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i,j} a_{ij}x_iy_j$ with respect to the standard basis is the matrix A whose (i,j)th entry is the coefficient of x_iy_j. In our case, we have

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}.$$

Let us now compute the eigenvectors of this matrix. The characteristic polynomial is

$$f(\lambda) = \lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 11\lambda + 24 = (\lambda - 3)(\lambda - 8),$$

so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 8$. The corresponding eigenvectors turn out to be

$$\boldsymbol{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Finally, we find the matrix M with respect to the basis $\boldsymbol{v}_1, \boldsymbol{v}_2$. Since $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^t A \boldsymbol{y}$, we get

$$m_{ij} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle = \boldsymbol{v}_i^t A \boldsymbol{v}_j = \lambda_j \boldsymbol{v}_i^t \boldsymbol{v}_j = \lambda_j (\boldsymbol{v}_i \cdot \boldsymbol{v}_j) \quad \Longrightarrow \quad M = \begin{bmatrix} 15 & 0 \\ 0 & 40 \end{bmatrix}.$$

Linear algebra II
Homework #7 solutions

1. Consider $\mathbb{R}^3$ with the usual dot product. Use the Gram-Schmidt procedure to find an orthogonal basis, starting with the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Using the Gram-Schmidt procedure, we let $\boldsymbol{w}_1 = \boldsymbol{v}_1$ and replace the second vector by

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \frac{\langle \boldsymbol{v}_2, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}.$$

As for the third vector $\boldsymbol{v}_3$, this needs to be replaced by

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - \frac{\langle \boldsymbol{v}_3, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 - \frac{\langle \boldsymbol{v}_3, \boldsymbol{w}_2 \rangle}{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle} \boldsymbol{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

2. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 6x_2 y_2.$$

Show that this is an inner product and use the Gram-Schmidt procedure to find an orthogonal basis for it, starting with the standard basis of $\mathbb{R}^2$.

The given form is symmetric because its matrix with respect to the standard basis is

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}.$$

To show that the form is also positive definite, we complete the square to find that

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = x_1^2 + 4x_1 x_2 + 6x_2^2 = (x_1 + 2x_2)^2 + 2x_2^2.$$

Finally, one may obtain an orthogonal basis by letting $\boldsymbol{w}_1 = \boldsymbol{e}_1$ and

$$\boldsymbol{w}_2 = \boldsymbol{e}_2 - \frac{\langle \boldsymbol{e}_2, \boldsymbol{w}_1 \rangle}{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle} \boldsymbol{w}_1 = \boldsymbol{e}_2 - \frac{\boldsymbol{e}_2^t A \boldsymbol{e}_1}{\boldsymbol{e}_1^t A \boldsymbol{e}_1} \boldsymbol{e}_1 = \boldsymbol{e}_2 - \frac{a_{21}}{a_{11}} \boldsymbol{e}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 1 & 3 \\ 4 & a \end{bmatrix}$. For which values of a is the form $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^t A \boldsymbol{y}$ positive definite?

Letting $\boldsymbol{y} = \boldsymbol{x}$ and completing the square, one finds that

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle = x_1^2 + 7x_1x_2 + ax_2^2 = (x_1 + 7x_2/2)^2 + (a - 49/4)x_2^2.$$

It easily follows that the given form is positive definite if and only if $a > 49/4$.

4. Let P_1 be the space of all real polynomials of degree at most 1 and let

$$\langle f, g \rangle = \int_0^1 (4 - 5x) \cdot f(x)g(x)\, dx \quad \text{for all } f, g \in P_1.$$

Is this bilinear form positive definite? Hint: compute $\langle ax + b, ax + b \rangle$ for all $a, b \in \mathbb{R}$.

Every element of P_1 has the form $f(x) = ax + b$ for some $a, b \in \mathbb{R}$ and this gives

$$\langle f, f \rangle = \langle ax + b, ax + b \rangle = \int_0^1 (4 - 5x) \cdot (ax + b)^2\, dx.$$

We now expand the quadratic factor and then integrate to get

$$\begin{aligned}
\langle f, f \rangle &= \int_0^1 (4 - 5x) \cdot (a^2x^2 + 2abx + b^2)\, dx \\
&= \int_0^1 \Big(4b^2 + (8ab - 5b^2)x + (4a^2 - 10ab)x^2 - 5a^2x^3 \Big) dx \\
&= 4b^2 + \frac{8ab - 5b^2}{2} + \frac{4a^2 - 10ab}{3} - \frac{5a^2}{4}.
\end{aligned}$$

To see that the form is positive definite, it remains to rearrange terms and write

$$\langle f, f \rangle = \frac{a^2 + 8ab + 18b^2}{12} = \frac{(a + 4b)^2 + 2b^2}{12}.$$

Linear algebra II
Homework #8 solutions

1. Find an orthogonal matrix B such that $B^t AB$ is diagonal when

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

The eigenvalues of A are the roots of the characteristic polynomial

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 9\lambda^2 - 18\lambda = -\lambda(\lambda - 3)(\lambda - 6).$$

This gives $\lambda_1 = 0$, $\lambda_2 = 3$ and $\lambda_3 = 6$, while the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Since the eigenvalues are distinct, the eigenvectors are orthogonal to one another. We may thus divide each of them by its length to obtain an orthogonal matrix B such that

$$B = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix} \quad \Longrightarrow \quad B^t AB = B^{-1}AB = \begin{bmatrix} 0 & & \\ & 3 & \\ & & 6 \end{bmatrix}.$$

2. Let P_1 be the space of all real polynomials of degree at most 1 and let

$$\langle f, g\rangle = \int_{-1}^{1} 3x \cdot f(x)g(x)\, dx \quad \text{for all } f, g \in P_1.$$

Find the matrix A of this bilinear form with respect to the standard basis and then find an orthogonal matrix B such that $B^t AB$ is diagonal.

By definition, the entries of the matrix A are given by the formula

$$a_{ij} = \langle x^{i-1}, x^{j-1}\rangle = \int_{-1}^{1} 3x^{i+j-1}\, dx.$$

This gives $a_{ij} = 0$ when $i + j$ is even and also $a_{ij} = 6/(i + j)$ when $i + j$ is odd, so

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = -2$, while the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Since the eigenvalues are distinct, the eigenvectors are orthogonal to one another. We may thus divide each of them by its length to obtain an orthogonal matrix B such that

$$B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \quad \Longrightarrow \quad B^t AB = B^{-1}AB = \begin{bmatrix} 2 & \\ & -2 \end{bmatrix}.$$

3. Show that every eigenvalue λ of a real orthogonal matrix B has absolute value 1. In other words, show that every eigenvalue λ of B satisfies $\lambda\overline{\lambda} = 1$.

Assuming that $\boldsymbol{v}$ is an eigenvector of B with eigenvalue λ, we get

$$\lambda\overline{\lambda}\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \langle \lambda\boldsymbol{v}, \lambda\boldsymbol{v}\rangle = \langle B\boldsymbol{v}, B\boldsymbol{v}\rangle = \langle B^*B\boldsymbol{v}, \boldsymbol{v}\rangle.$$

Since B is real and orthogonal, one has $B^*B = B^tB = I_n$ and this implies that

$$\lambda\overline{\lambda}\langle \boldsymbol{v}, \boldsymbol{v}\rangle = \langle B^*B\boldsymbol{v}, \boldsymbol{v}\rangle = \langle \boldsymbol{v}, \boldsymbol{v}\rangle \quad \Longrightarrow \quad \lambda\overline{\lambda} = 1.$$

4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form an orthonormal basis of $\mathbb{R}^n$ and consider the $n \times n$ matrix $A = I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t$. Show that A is symmetric, orthogonal and diagonalisable.

To say that A is symmetric is to say that $A^t = A$ and this is true because

$$A^t = I_n^t - 2(\boldsymbol{v}_1\boldsymbol{v}_1^t)^t = I_n - 2\boldsymbol{v}_1^{tt}\boldsymbol{v}_1^t = I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t = A.$$

To say that A is orthogonal is to say that $A^tA = I_n$ and this is true because

$$A^tA = AA = (I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t)(I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t) = I_n - 4\boldsymbol{v}_1\boldsymbol{v}_1^t + 4\boldsymbol{v}_1(\boldsymbol{v}_1^t\boldsymbol{v}_1)\boldsymbol{v}_1^t = I_n.$$

Finally, we show that A is diagonalisable. This follows by the spectral theorem, but it can also be verified directly by showing that each $\boldsymbol{v}_i$ is an eigenvector of A. In fact, one has

$$\begin{aligned} A\boldsymbol{v}_1 &= \boldsymbol{v}_1 - 2\boldsymbol{v}_1(\boldsymbol{v}_1^t\boldsymbol{v}_1) = -\boldsymbol{v}_1, \\ A\boldsymbol{v}_k &= \boldsymbol{v}_k - 2\boldsymbol{v}_1(\boldsymbol{v}_1^t\boldsymbol{v}_k) = \boldsymbol{v}_k \end{aligned}$$

for each $2 \leq k \leq n$. This gives n linearly independent eigenvectors, so A is diagonalisable.