Linear algebra II
Homework #1
due Thursday, Feb. 2

1. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & 6 \\ 2 & 5 \end{bmatrix}.$$

2. Find the eigenvalues and the eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 1 & -1 & 1 \\ 3 & -9 & 5 \end{bmatrix}.$$

3. The following matrix has eigenvalues $\lambda = 1, 1, 1, 4$. Is it diagonalisable? Explain.

$$A = \begin{bmatrix} 3 & 1 & -3 & 3 \\ 2 & 2 & -3 & 3 \\ 2 & 1 & -2 & 3 \\ 3 & 0 & -3 & 4 \end{bmatrix}.$$

4. Suppose A is a square matrix which is both diagonalisable and invertible. Show that every eigenvector of A is an eigenvector of A^{-1} and that A^{-1} is diagonalisable.

- When writing up solutions, write legibly and coherently.
- Write your name and then MATHS/TP/TSM on the first page of your homework.
- NO LATE HOMEWORK WILL BE ACCEPTED.

Linear algebra II
Homework #2
due Thursday, Feb. 9

1. Find a basis for both the null space and the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 5 \\ 3 & 1 & 5 & 0 \\ 4 & 2 & 6 & 2 \\ 4 & 3 & 5 & 5 \end{bmatrix}.$$

2. Show that the following matrix is diagonalisable.

$$A = \begin{bmatrix} 7 & 1 & -3 \\ 3 & 5 & -3 \\ 5 & 1 & -1 \end{bmatrix}.$$

3. Find the eigenvalues and the generalised eigenvectors of the matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ -3 & 1 & 3 \\ -5 & 1 & 6 \end{bmatrix}.$$

4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ form a basis for the null space of a square matrix A and that $C = B^{-1}AB$ for some invertible matrix B. Find a basis for the null space of C.

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Linear algebra II
Homework #3
due Thursday, Feb. 16

1. The following matrix has $\lambda = 4$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 1 & 5 & 1 \\ -2 & 7 & 1 \\ -1 & 2 & 4 \end{bmatrix}.$$

2. The following matrix has $\lambda = 4$ as its only eigenvalue. What is its Jordan form?

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 6 & 2 \\ -1 & 1 & 5 \end{bmatrix}.$$

3. Find a Jordan chain of length 2 for the matrix

$$A = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix}.$$

4. Let A be a 3×3 matrix that has $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ as a Jordan chain of length 3 and let B be the matrix whose columns are $\boldsymbol{v}_3, \boldsymbol{v}_2, \boldsymbol{v}_1$ (in the order listed). Compute $B^{-1}AB$.

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Linear algebra II
Homework #4
due Thursday, Feb. 23

1. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -2 & 6 \\ 2 & -1 & 3 \end{bmatrix}.$$

2. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} 2 & 4 & -2 \\ 1 & 4 & -1 \\ 2 & 0 & 2 \end{bmatrix}.$$

3. Suppose that A is a 2×2 matrix with $A^2 = A$. Show that A is diagonalisable.

4. A 5×5 matrix A has characteristic polynomial $f(\lambda) = \lambda^4(2 - \lambda)$ and its column space is two-dimensional. Find the dimension of the column space of A^2.

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Linear algebra II
Homework #5
due Thursday, March 9

1. Let $x_0 = 1$ and $y_0 = -2$. Suppose the sequences x_n, y_n are such that

$$x_n = 4x_{n-1} - y_{n-1}, \qquad y_n = x_{n-1} + 2y_{n-1}$$

for each $n \geq 1$. Determine each of x_n and y_n explicitly in terms of n.

2. Which of the following matrices are similar? Explain.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

3. Let J be a 4×4 Jordan block with eigenvalue $\lambda = 0$. Find the Jordan form of J^2.

4. Use the Cayley-Hamilton theorem to compute A^{2017} in the case that

$$A = \begin{bmatrix} 3 & -2 & -3 \\ 2 & -2 & -2 \\ 3 & -2 & -3 \end{bmatrix}.$$

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Linear algebra II
Homework #6
due Thursday, March 16

1. The following matrix has $\lambda = -1$ as a triple eigenvalue. Use this fact to find its Jordan form, its minimal polynomial and also its inverse.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 6 & -4 & 3 \\ 2 & -1 & 0 \end{bmatrix}.$$

2. The following matrix has eigenvalues $\lambda = 0, 1, 1$. Use this fact to find its Jordan form, its minimal polynomial and also its power A^{2017}.

$$A = \begin{bmatrix} 4 & -5 & 2 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

3. Let P_2 be the space of all real polynomials of degree at most 2 and let

$$\langle f, g \rangle = \int_0^1 (3 - x) \cdot f(x) g(x)\, dx \quad \text{for all } f, g \in P_2.$$

Find the matrix of this bilinear form with respect to the standard basis.

4. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 4x_1y_1 + 2x_1y_2 + 2x_2y_1 + 7x_2y_2.$$

Find the matrix A of this form with respect to the standard basis and then find the matrix with respect to a basis consisting of eigenvectors of A.

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Linear algebra II
Homework #7
due Thursday, March 23

1. Consider $\mathbb{R}^3$ with the usual dot product. Use the Gram-Schmidt procedure to find an orthogonal basis, starting with the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

2. Define a bilinear form on $\mathbb{R}^2$ by setting

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + 2x_1 y_2 + 2x_2 y_1 + 6x_2 y_2.$$

Show that this is an inner product and use the Gram-Schmidt procedure to find an orthogonal basis for it, starting with the standard basis of $\mathbb{R}^2$.

3. Let $A = \begin{bmatrix} 1 & 3 \\ 4 & a \end{bmatrix}$. For which values of a is the form $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^t A \boldsymbol{y}$ positive definite?

4. Let P_1 be the space of all real polynomials of degree at most 1 and let

$$\langle f, g \rangle = \int_0^1 (4 - 5x) \cdot f(x) g(x)\, dx \quad \text{for all } f, g \in P_1.$$

Is this bilinear form positive definite? Hint: compute $\langle ax + b, ax + b \rangle$ for all $a, b \in \mathbb{R}$.

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Linear algebra II
Homework #8
due Thursday, March 30

1. Find an orthogonal matrix B such that $B^t AB$ is diagonal when

$$A = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

2. Let P_1 be the space of all real polynomials of degree at most 1 and let

$$\langle f, g\rangle = \int_{-1}^{1} 3x \cdot f(x)g(x)\, dx \quad \text{for all } f, g \in P_1.$$

Find the matrix A of this bilinear form with respect to the standard basis and then find an orthogonal matrix B such that $B^t AB$ is diagonal.

3. Show that every eigenvalue λ of a real orthogonal matrix B has absolute value 1. In other words, show that every eigenvalue λ of B satisfies $\lambda\overline{\lambda} = 1$.

4. Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ form an orthonormal basis of $\mathbb{R}^n$ and consider the $n \times n$ matrix $A = I_n - 2\boldsymbol{v}_1\boldsymbol{v}_1^t$. Show that A is symmetric, orthogonal and diagonalisable.

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