Linear algebra II
2016 exam solutions

1. Find a matrix A that has $\boldsymbol{v}_1$ as an eigenvector with eigenvalue $\lambda_1 = 4$ and $\boldsymbol{v}_2$ as an eigenvector with eigenvalue $\lambda_2 = 5$ when

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

If B is the matrix whose columns are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, then the general theory implies that

$$B^{-1}AB = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & \\ & 5 \end{bmatrix}.$$

Once we now solve this equation for A, we may conclude that

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & \\ & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = - \begin{bmatrix} 4 & 10 \\ 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 1 & 3 \end{bmatrix}.$$

2. Find the Jordan form and a Jordan basis for the matrix

$$A = \begin{bmatrix} 4 & -7 & 5 \\ 1 & -3 & 4 \\ 1 & -6 & 7 \end{bmatrix}.$$

The characteristic polynomial of the given matrix is

$$f(\lambda) = \det(A - \lambda I) = -\lambda^3 + 8\lambda^2 - 21\lambda + 18 = (2 - \lambda)(\lambda - 3)^2.$$

Thus, the eigenvalues are $\lambda = 2, 3, 3$ and one can easily determine the null spaces

$$\mathcal{N}(A - 2I) = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \qquad \mathcal{N}(A - 3I) = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

This implies that A is not diagonalisable and that its Jordan form is

$$J = B^{-1}AB = \left[\begin{array}{c|cc} 2 & & \\ \hline & 3 & \\ & 1 & 3 \end{array}\right].$$

To find a Jordan basis, we need to find vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ such that $\boldsymbol{v}_1$ is an eigenvector with eigenvalue $\lambda = 2$ and $\boldsymbol{v}_2, \boldsymbol{v}_3$ is a Jordan chain with eigenvalue $\lambda = 3$. In our case, we have

$$\mathcal{N}(A - 3I)^2 = \operatorname{Span}\left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\},$$

so it easily follows that a Jordan basis is provided by the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_3 = (A - 3I)\boldsymbol{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}.$$

3. The following matrix A has a triple eigenvalue. Find the minimal polynomial of A and use it to determine the inverse of A.

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & -2 & 2 \\ 3 & -6 & 5 \end{bmatrix}.$$

There is a triple eigenvalue λ and the sum of the eigenvalues is the trace of A, so

$$3\lambda = \operatorname{tr} A = 3 - 2 + 5 = 6 \quad \Longrightarrow \quad \lambda = 2.$$

The number of Jordan blocks is the dimension of $\mathcal{N}(A - 2I)$ and row reduction gives

$$A - 2I = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ 3 & -6 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, there are two free variables and two Jordan blocks, so there is a 2×2 block as well as an 1×1 block. Since the largest block is 2×2, the minimal polynomial is

$$m(\lambda) = (\lambda - 2)^2 = \lambda^2 - 4\lambda + 4.$$

The given matrix must obviously satisfy this polynomial, so

$$4I = 4A - A^2 = A(4I - A) \quad \Longrightarrow \quad A^{-1} = \frac{1}{4}(4I - A) = \frac{1}{4}\begin{bmatrix} 1 & 2 & -1 \\ -2 & 6 & -2 \\ -3 & 6 & -1 \end{bmatrix}.$$

4. Let Q be the quadratic form on $\mathbb{R}^3$ which is defined by the formula

$$Q(x, y, z) = 2x^2 + (a + 4)y^2 + (a + 4)z^2 + 2axy + 6axz.$$

Find the values of the real parameter a for which the form is positive definite.

The given quadratic can be expressed in the form $Q(\boldsymbol{x}) = \boldsymbol{x}^t A\boldsymbol{x}$, where

$$A = \begin{bmatrix} 2 & a & 3a \\ a & a + 4 & 0 \\ 3a & 0 & a + 4 \end{bmatrix}.$$

According to Sylvester's criterion, this matrix is positive definite if and only if

$$\det \begin{bmatrix} 2 & a \\ a & a+4 \end{bmatrix} = 2a + 8 - a^2, \qquad \det A = -10a^3 - 38a^2 + 16a + 32$$

are both positive. When it comes to the first determinant, one finds that

$$2a + 8 - a^2 > 0 \quad \Longleftrightarrow \quad (a+2)(a-4) < 0 \quad \Longleftrightarrow \quad -2 < a < 4.$$

When it comes to the second determinant, one similarly finds that

$$\det A = -10a^3 - 38a^2 + 16a + 32 = -2(a-1)(a+4)(5a+4).$$

It easily follows that A is positive definite if and only if $-4/5 < a < 1$.

5. Suppose that A is an invertible $n \times n$ real matrix. Show that there exists a positive definite symmetric matrix P such that $P^2 = A^t A$.

The matrix $A^t A$ is symmetric because $(A^t A)^t = A^t A^{tt} = A^t A$ and positive definite, as

$$\boldsymbol{x}^t (A^t A) \boldsymbol{x} = (A\boldsymbol{x})^t (A\boldsymbol{x}) = ||A\boldsymbol{x}||^2 > 0$$

for all $\boldsymbol{x} \neq 0$. It follows by the spectral theorem that there exists an orthogonal matrix B such that $D = B^t (A^t A) B$ is diagonal. The diagonal entries of D are the eigenvalues λ_i of the matrix $A^t A$ and those are all positive. Let us denote by C the diagonal matrix whose diagonal entries are $\sqrt{\lambda_i}$. Then C is symmetric with $C^2 = D$ and we also have

$$(BCB^t)^2 = BCB^t \cdot BCB^t = BC^2 B^t = BDB^t = A^t A.$$

In particular, the matrix $P = BCB^t$ satisfies the given condition. This matrix is symmetric because $P^t = B^{tt} C^t B^t = BCB^t = P$ and it is positive definite because its eigenvalues are the same as the eigenvalues of C, so they are all positive.