# MA121 Tutorial Problems #8
## Solutions

**1.** *Classify the critical points of* $f(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2$.

- To find the critical points, we need to solve the equations

$$0 = f_x(x,y) = 4x^3 + 8xy^2 - 4x = 4x(x^2 + 2y^2 - 1),$$
$$0 = f_y(x,y) = 8x^2y + 4y = 4y(2x^2 + 1).$$

Since the second equation gives $y = 0$, the first equation becomes

$$0 = 4x(x^2 - 1) = 4x(x+1)(x-1) \quad \Longrightarrow \quad x = 0, -1, 1.$$

Thus, there are three critical points and those are $(0,0)$, $(-1,0)$ and $(1,0)$.

- In order to classify the critical points, we compute the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12x^2 + 8y^2 - 4 & 16xy \\ 16xy & 8x^2 + 4 \end{bmatrix}.$$

When it comes to the critical point $(0,0)$, this gives

$$H = \begin{bmatrix} -4 & 0 \\ 0 & 4 \end{bmatrix} \quad \Longrightarrow \quad \det H = -16 < 0$$

so the origin is a saddle point. When it comes to the critical points $(\pm 1, 0)$, we have

$$H = \begin{bmatrix} 8 & 0 \\ 0 & 12 \end{bmatrix} \quad \Longrightarrow \quad \det H = 8 \cdot 12 > 0$$

and also $f_{xx} = 8 > 0$, so each of these points is a local minimum.

**2.** *Classify the critical points of* $f(x,y) = 3x^2 - 3y^2 + 8xy + 10x - 20y + 30$.

- To find the critical points, we need to solve the equations

$$0 = f_x(x,y) = 6x + 8y + 10, \qquad 0 = f_y(x,y) = -6y + 8x - 20.$$

We multiply the first equation by 6, the second one by 8 and then we add to get

$$0 = 36x + 60 + 64x - 160 = 100x - 100 \quad \Longrightarrow \quad x = 1.$$

The first equation now gives $8y = -16$, so the only critical point is $(1, -2)$.

- In order to classify this critical point, we compute the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}.$$

Since $\det H = -36 - 64$ is negative, the critical point is actually a saddle point.

**3.** *Compute each of the following integrals:*

$$\int_0^1\int_x^1 \cos(x/y)\,dy\,dx, \qquad \int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy, \qquad \int_0^6\int_{x/3}^2 x\sqrt{y^3+1}\;dy\,dx.$$

- For the first integral, switching the order of integration gives

$$\int_0^1\int_x^1 \cos(x/y)\,dy\,dx = \int_0^1\int_0^y \cos(x/y)\,dx\,dy$$

and one can easily compute the inner integral, namely

$$\int_0^y \cos(x/y)\,dx = \Big[y\sin(x/y)\Big]_{x=0}^{x=y} = y\sin 1.$$

In particular, the first double integral is equal to

$$\int_0^1\int_x^1 \cos(x/y)\,dy\,dx = \int_0^1 y\sin 1\,dy = \frac{\sin 1}{2}\,.$$

- For the second integral, switching the order of integration gives

$$\int_0^1\int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3\int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 \frac{xe^{x^2}}{3}\,dx = \left[\frac{e^{x^2}}{6}\right]_0^3 = \frac{e^9-1}{6}\,.$$

- For the third integral, switching the order of integration gives

$$\int_0^6\int_{x/3}^2 x\sqrt{y^3+1}\;dy\,dx = \int_0^2\int_0^{3y} x\sqrt{y^3+1}\;dx\,dy = \int_0^2 \frac{9y^2\sqrt{y^3+1}}{2}\,dy$$

and one can now use the substitution $u = y^3+1$ to conclude that

$$\int_0^6\int_{x/3}^2 x\sqrt{y^3+1}\,dy\,dx = \int_1^9 \frac{3\sqrt{u}}{2}\,du = \Big[u^{3/2}\Big]_1^9 = 26.$$