MA121 Tutorial Problems #7
Solutions

1. *Letting $f(x,y) = x^2e^y + xy$, find the direction in which f increases most rapidly at the point $(2,0)$. What is the exact rate of change in that direction?*

- The direction of most rapid increase is given by the gradient vector

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \langle 2xe^y + y, x^2e^y + x \rangle \quad \Longrightarrow \quad \nabla f(2,0) = \langle 4, 6 \rangle.$$

To find the exact rate of change in that direction, we first find a unit vector $\mathbf{u}$ in the same direction. This amounts to dividing the vector $\nabla f(2,0)$ by its length, namely

$$||\nabla f(2,0)|| = \sqrt{4^2 + 6^2} = \sqrt{52} \quad \Longrightarrow \quad \mathbf{u} = \langle 4/\sqrt{52}, 6/\sqrt{52}\, \rangle.$$

The desired rate of change is then given by the directional derivative

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{4 \cdot 4}{\sqrt{52}} + \frac{6 \cdot 6}{\sqrt{52}} = \frac{52}{\sqrt{52}} = \sqrt{52} = 2\sqrt{13}.$$

2. *Assuming that $z = f(x^2 - y^2, y^2 - x^2)$ for some differentiable function f, show that*

$$yz_x + xz_y = 0.$$

- In this case, $z = f(u,v)$ with $u = x^2 - y^2$ and $v = y^2 - x^2$, so the chain rule gives

$$\begin{aligned} z_x &= f_x = f_u u_x + f_v v_x = 2xf_u - 2xf_v, \\ z_y &= f_y = f_u u_y + f_v v_y = -2yf_u + 2yf_v. \end{aligned}$$

Once we now combine these two equations, we get the desired identity

$$yz_x + xz_y = 2xyf_u - 2xyf_v - 2xyf_u + 2xyf_v = 0.$$

3. *Find the minimum value of $f(x,y) = 2x^2 - 4x + 3y^2$ over the closed disk*

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}.$$

- To find the critical points, we need to solve the equations

$$0 = f_x(x,y) = 4x - 4 = 4(x-1), \qquad 0 = f_y(x,y) = 6y.$$

Thus, the only critical point is $(1,0)$ and this corresponds to the value $f(1,0) = -2$.

- Next, we check the points on the boundary of the disk. Along the boundary,

$$y^2 = 9 - x^2 \quad \Longrightarrow \quad f(x,y) = 2x^2 - 4x + 3(9 - x^2) = -x^2 - 4x + 27$$

  and we need to find the minimum value of this function on $[-3, 3]$. Noting that

$$g(x) = -x^2 - 4x + 27 \quad \Longrightarrow \quad g'(x) = -2x - 4 = -2(x+2),$$

  we see that the minimum value may only occur at $x = -3$, $x = 3$ or $x = -2$. Since

$$g(-3) = 30, \qquad g(3) = 6, \qquad g(-2) = 31,$$

  the smallest value we have found so far is the value $f(1,0) = -2$ obtained above.

4. *Find the maximum value of* $f(x,y) = x^2 + xy + 3x + 2y$ *over the region*

$$R = \{(x,y) \in \mathbb{R}^2 : x^2 \leq y \leq 4\}.$$

- To find the critical points, we need to solve the equations

$$0 = f_x(x,y) = 2x + y + 3, \qquad 0 = f_y(x,y) = x + 2.$$

  Since $x = -2$ by the second equation, we get $y = -2x - 3 = 1$ by the first equation. In particular, $(-2, 1)$ is the only critical point, however this point does not lie in the given region, so we may simply ignore it.

- Next, we check the points on the boundary. Along the parabola $y = x^2$, we have

$$f(x,y) = x^2 + x^3 + 3x + 2x^2 = x^3 + 3x^2 + 3x$$

  and we need to find the maximum value of this function on $[-2, 2]$. Noting that

$$g(x) = x^3 + 3x^2 + 3x \quad \Longrightarrow \quad g'(x) = 3(x^2 + 2x + 1) = 3(x+1)^2,$$

  we see that the maximum value may only occur at $x = -2$, $x = 2$ or $x = -1$. Since

$$g(-2) = -2, \qquad g(2) = 26, \qquad g(-1) = -1,$$

  the maximum value is $g(2) = 26$ and this corresponds to the value $f(2,4) = 26$.

- It now remains to check the boundary points along the line $y = 4$. For these points,

$$f(x,y) = x^2 + 4x + 3x + 8 = x^2 + 7x + 8$$

  and we need to find the maximum value of this function on $[-2, 2]$. Noting that

$$h(x) = x^2 + 7x + 8 \quad \Longrightarrow \quad h'(x) = 2x + 7 = 2(x + 7/2),$$

  we see that the maximum value may only occur at $x = -2$ or $x = 2$. Since

$$h(-2) = 4 - 14 + 8 = -2, \qquad h(2) = 4 + 14 + 8 = 26,$$

  the maximum value over the whole region is the value $f(2,4) = 26$ obtained above.

5. *Find the maximum value of* $f(x,y) = x^3 + y^3 - 3xy$ *over the closed triangular region whose vertices are the points* $(0,0)$, $(1,0)$ *and* $(1,2)$.

- To find the critical points, we need to solve the equations

$$0 = f_x(x,y) = 3x^2 - 3y = 3(x^2 - y),$$
$$0 = f_y(x,y) = 3y^2 - 3x = 3(y^2 - x).$$

These give $y = x^2$ and also $x = y^2$, so we easily get

$$x = y^2 = x^4 \quad \Longrightarrow \quad x^4 - x = 0 \quad \Longrightarrow \quad x(x^3 - 1) = 0 \quad \Longrightarrow \quad x = 0, 1.$$

Thus, the only critical points are $(0,0)$ and $(1,1)$, while $f(0,0) = 0$ and $f(1,1) = -1$.

- Next, we check the points on the boundary of the region. Along the horizontal side,

$$y = 0 \quad \Longrightarrow \quad f(x,y) = x^3$$

and we have $0 \leq x \leq 1$, so the maximum value is $f(1,0) = 1$. Along the vertical side,

$$x = 1 \quad \Longrightarrow \quad f(x,y) = y^3 - 3y + 1$$

and we need to find the maximum value of this function on $[0,2]$. Noting that

$$g(y) = y^3 - 3y + 1 \quad \Longrightarrow \quad g'(y) = 3y^2 - 3 = 3(y^2 - 1),$$

we see that the maximum value may only occur at $y = 0$, $y = 2$ or $y = 1$. Since

$$g(0) = 1, \qquad g(2) = 8 - 6 + 1 = 3, \qquad g(1) = 1 - 3 + 1 = -1,$$

the largest value so far is the value $g(2) = 3$ corresponding to $f(1,2) = 3$. It remains to check the boundary points along the hypotenuse. For these points,

$$y = 2x \quad \Longrightarrow \quad f(x,y) = x^3 + (2x)^3 - 3x(2x) = 9x^3 - 6x^2$$

and we need to find the maximum value of this function on $[0,1]$. Noting that

$$h(x) = 9x^3 - 6x^2 \quad \Longrightarrow \quad h'(x) = 27x^2 - 12x = 3x(9x - 4),$$

we see that the maximum value may only occur at $x = 0$, $x = 1$ or $x = 4/9$. Since

$$h(0) = 0, \qquad h(1) = 9 - 6 = 3, \qquad h(4/9) = -32/81,$$

the maximum value over the whole triangular region is the value $f(1,2) = 3$.