MA121 Tutorial Problems #6
Solutions

1. *Find the radius of convergence for each of the following power series:*

$$\sum_{n=0}^{\infty} \frac{nx^n}{3^n}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n\, x^n}{2n+1}, \qquad \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \cdot x^n.$$

- One always uses the ratio test to find the radius of convergence. In the first case,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|x|^{n+1}}{|x|^n} \cdot \frac{3^n}{3^{n+1}} = \frac{|x|}{3}$$

so the series converges when $|x|/3 < 1$ and diverges when $|x|/3 > 1$. Thus, the series converges when $|x| < 3$ and diverges when $|x| > 3$; this also means that $R = 3$.

- When it comes to the second series, a similar computation gives

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{2n+1}{2n+3} = |x|$$

and it easily follows that the radius of convergence is $R = 1$.

- When it comes to the last series, finally, we have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{x^{n+1}}{x^n} = \frac{(n+1)^2 \cdot x}{(2n+1)(2n+2)}$$

and this implies that

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n^2+2n+1)\,|x|}{4n^2+6n+2} = \frac{|x|}{4}.$$

Thus, the series converges when $|x| < 4$ and diverges when $|x| > 4$ so that $R = 4$.

2. *Although a power series may be differentiated term by term, this is not really the case for an arbitrary series. In fact, an infinite sum of continuous/differentiable functions does not even have to be continuous/differentiable itself. To see this, let*

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \qquad f_n(x) = \frac{x^2}{(1+x^2)^n}$$

and check that f is not continuous at $x = 0$, even though each f_n is.

- Each f_n is a rational function which is defined at all points, so each f_n is continuous at all points. To show that f is not continuous, let us first recall the formula

$$\sum_{n=0}^{\infty} y^n = \frac{1}{1-y}$$

which is valid whenever $|y| < 1$. Using this formula, one easily finds that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \frac{x^2}{1 - \frac{1}{1+x^2}} = 1 + x^2$$

whenever $\frac{1}{1+x^2} < 1$. This gives $f(x) = 1 + x^2$ whenever $x \neq 0$, and we also have

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} \quad \Longrightarrow \quad f(0) = 0.$$

In particular, f is not continuous at $x = 0$ because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (1 + x^2) = 1 \neq f(0).$$

3. *Consider the function f defined by the power series*

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(a) *Show that f is defined for all $x \in \mathbb{R}$ and that we also have*

$$f'(x) = f(x), \qquad f(0) = 1.$$

(b) *Use part (a) to show that $f(x)f(-x) = 1$ and that $f(x) > 0$ for all $x \in \mathbb{R}$.*

- To show that the given series converges for all x, we use the ratio test. Since

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|x|^{n+1}}{|x|^n} \cdot \frac{n!}{(n+1)!} = \lim_{n \to \infty} \frac{|x|}{n+1} = 0,$$

we have $L < 1$ for any x whatsoever, so the series converges for any x whatsoever.

- To show that $f'(x) = f(x)$, we differentiate the series term by term to get

$$f'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \ldots = 1 + x + \frac{x^2}{2!} + \ldots = f(x).$$

- To show that $f(x)f(-x) = 1$, we let $g(x) = f(x)f(-x)$ and we note that

$$\begin{aligned} g'(x) &= f'(x) \cdot f(-x) + f(x) \cdot f'(-x) \cdot (-x)' \\ &= f(x) \cdot f(-x) - f(x) \cdot f(-x) \\ &= 0. \end{aligned}$$

  In particular, $g(x)$ is constant and we have $g(x) = g(0) = f(0)^2 = 1$.

- Since $f(x)f(-x) = 1$ by above, f can never be zero. According to Bolzano's theorem then, f is either positive at all points or else negative at all points. Since $f(0) = 1$, this means that f must actually be positive at all points.