MA121 Tutorial Problems #5
Solutions

1. *Let $x > 0$ and define a sequence $\{a_n\}$ by picking any number $a_1 \geq \sqrt{x}$ and by setting*

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) \quad \textit{for each } n \geq 1.$$

Show that $a_n \geq \sqrt{x}$ for all n, that $\{a_n\}$ is decreasing and that $\lim_{n\to\infty} a_n = \sqrt{x}$.

- First of all, note that $a_1 \geq \sqrt{x}$ by assumption and that

$$\begin{aligned} a_{n+1} \geq \sqrt{x} \quad &\Longleftrightarrow \quad a_n + \frac{x}{a_n} \geq 2\sqrt{x} \quad \Longleftrightarrow \quad a_n^2 + x \geq 2a_n\sqrt{x} \\ &\Longleftrightarrow \quad (a_n - \sqrt{x}\,)^2 \geq 0. \end{aligned}$$

Since the last inequality is clearly true, the first inequality is also true and so $a_n \geq \sqrt{x}$ for all n. To show that $\{a_n\}$ is decreasing, we note that

$$\begin{aligned} a_{n+1} \leq a_n \quad &\Longleftrightarrow \quad a_n + \frac{x}{a_n} \leq 2a_n \quad \Longleftrightarrow \quad a_n^2 + x \leq 2a_n^2 \\ &\Longleftrightarrow \quad \sqrt{x} \leq a_n. \end{aligned}$$

Since the last inequality is true by above, the first inequality is also true, so the given sequence is both monotonic and bounded. Letting L denote its limit, we now get

$$\begin{aligned} 2a_{n+1} = a_n + \frac{x}{a_n} \quad &\Longrightarrow \quad 2L = L + \frac{x}{L} \quad \Longrightarrow \quad 2L^2 = L^2 + x \\ &\Longrightarrow \quad L = \pm\sqrt{x}. \end{aligned}$$

Since $a_n \geq \sqrt{x}$ for all n, however, we also have $L \geq \sqrt{x}$ and this implies $L = \sqrt{x}$.

2. *Compute each of the following sums:*

$$\sum_{n=0}^{\infty} \frac{2^n}{7^n}, \qquad \sum_{n=0}^{\infty} \frac{3^{n+2}}{2^{2n+1}}, \qquad \sum_{n=1}^{\infty} \frac{5^{n+1}}{2^{3n}}.$$

- When it comes to the first sum, the formula for a geometric series gives

$$\sum_{n=0}^{\infty} \frac{2^n}{7^n} = \frac{1}{1-2/7} = \frac{1}{5/7} = \frac{7}{5}.$$

- When it comes to the second sum, a similar computation gives

$$\sum_{n=0}^{\infty} \frac{3^{n+2}}{2^{2n+1}} = \frac{3^2}{2^1} \cdot \sum_{n=0}^{\infty} \frac{3^n}{4^n} = \frac{9}{2} \cdot \frac{1}{1-3/4} = \frac{9}{2} \cdot 4 = 18.$$

- As for the third sum, this is a geometric series without its $n = 0$ term, namely

$$\sum_{n=1}^{\infty} \frac{5^{n+1}}{2^{3n}} = 5 \cdot \sum_{n=1}^{\infty} \frac{5^n}{8^n} = 5 \cdot \left(\frac{1}{1 - 5/8} - 1 \right) = \frac{25}{3}\,.$$

3. *Show that the sequence defined by $a_n = n \sin(1/n)$ is convergent.*

- We need to compute the limit of a_n as $n \to \infty$. Noting that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n}$$

is a $0/0$ limit, we may then apply L'Hôpital's rule to get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cos(1/n) \cdot \cancel{(1/n)'}}{\cancel{(1/n)'}} = \cos 0 = 1.$$

4. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} n \sin(1/n)\,, \qquad \sum_{n=1}^{\infty} \frac{1}{n + e^n}\,, \qquad \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}\,, \qquad \sum_{n=1}^{\infty} \frac{\log n}{n}\,.$$

- When it comes to the first series, the previous problem gives

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin(1/n) = 1$$

and so the first series diverges by the nth term test.

- To test the second series for convergence, we use the comparison test. Since

$$\sum_{n=1}^{\infty} \frac{1}{n + e^n} \leq \sum_{n=1}^{\infty} \frac{1}{e^n}\,,$$

the leftmost series is smaller than a convergent geometric series, so it converges.

- When it comes to the third series, we have $1/n \leq 1$ for all n, hence

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2} \leq \sum_{n=1}^{\infty} \frac{e}{n^2}\,.$$

Being smaller than a convergent p-series, the third series must thus be convergent.

- When it comes to the last series, finally, the fact that $\log 1 = 0$ gives

$$\sum_{n=1}^{\infty} \frac{\log n}{n} = \sum_{n=2}^{\infty} \frac{\log n}{n} \geq \sum_{n=2}^{\infty} \frac{\log 2}{n}\,.$$

Being bigger than a divergent p-series, the last series must thus be divergent.