MA121 Tutorial Problems #4
Solutions

1. *Suppose f is continuous on $[a,b]$. Show that there exists some $c \in (a,b)$ such that*

$$\int_a^b f(t)\,dt = (b-a)\cdot f(c).$$

As a hint, apply the mean value theorem to the function $F(x) = \int_a^x f(t)\,dt$.

- According to the mean value theorem, there exists some $c \in (a,b)$ such that

$$\frac{F(b)-F(a)}{b-a} = F'(c).$$

Moreover, $F'(x) = f(x)$ by the fundamental theorem of calculus, while

$$F(a) = \int_a^a f(t)\,dt = 0, \qquad F(b) = \int_a^b f(t)\,dt.$$

Once we now combine all these facts, we may conclude that

$$F(b) - F(a) = (b-a)\cdot F'(c) \quad\Longrightarrow\quad \int_a^b f(t)\,dt = (b-a)\cdot f(c).$$

2. *Suppose that $\sin x$ and $\cos x$ are two functions with the following properties:*

$$(\sin x)' = \cos x, \qquad (\cos x)' = -\sin x, \qquad \sin 0 = 0, \qquad \cos 0 = 1.$$

Show that $(\sin x)^2 + (\cos x)^2$ must be constant, namely $(\sin x)^2 + (\cos x)^2 = 1$.

- Letting $f(x) = \sin^2 x + \cos^2 x$ for convenience, one easily finds that

$$f'(x) = 2\sin x(\sin x)' + 2\cos x(\cos x)' = 2\sin x\cos x - 2\cos x\sin x = 0.$$

In particular, $f(x)$ must be constant and thus $f(x) = f(0) = \sin^2 0 + \cos^2 0 = 1$.

3. *Compute each of the following integrals:*

$$\int \frac{\sin(1/x)}{x^2}\,dx, \qquad \int (x+1)(x+2)^5\,dx, \qquad \int \frac{x}{\sqrt{x+1}}\,dx, \qquad \int xe^x\,dx.$$

- For the first integral, the substitution $u = 1/x = x^{-1}$ gives $du = -x^{-2}\,dx$ so that

$$\int \frac{\sin(1/x)}{x^2}\,dx = -\int \sin u\,du = \cos u + C = \cos(1/x) + C.$$

- For the second integral, set $u = x + 2$. This gives $du = dx$ and $x + 1 = u - 1$, so

$$\int (x+1)(x+2)^5 \, dx = \int (u-1)\, u^5 \, du = \int (u^6 - u^5) \, du$$
$$= \frac{u^7}{7} - \frac{u^6}{6} + C = \frac{(x+2)^7}{7} - \frac{(x+2)^6}{6} + C.$$

- For the third integral, set $u = x + 1$. This gives $du = dx$ and $x = u - 1$ so that

$$\int \frac{x}{\sqrt{x+1}} \, dx = \int \frac{u-1}{\sqrt{u}} \, du = \int \left(u^{1/2} - u^{-1/2}\right) du$$
$$= \frac{2u^{3/2}}{3} - 2u^{1/2} + C = \frac{2(x+1)^{3/2}}{3} - 2(x+1)^{1/2} + C.$$

- To compute the last integral, we shall use tabular integration.

Differentiating	Integrating
x	e^x
1	e^x
0	e^x

Consulting the table above, we find that

$$\int xe^x \, dx = xe^x - e^x + C.$$

4. *Compute each of the following integrals:*

$$\int \sin^3 x \, dx, \qquad \int \frac{x}{e^x} \, dx, \qquad \int e^{\sqrt{x}} \, dx, \qquad \int \frac{\log x}{x^2} \, dx.$$

- For the first integral, we set $u = \cos x$. This gives $du = -\sin x \, dx$, hence also

$$\int \sin^3 x \, dx = \int (1 - \cos^2 x) \sin x \, dx = \int (u^2 - 1) \, du$$
$$= \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C.$$

- To compute the second integral, we shall use tabular integration.

Differentiating	Integrating
x	e^{-x}
1	$-e^{-x}$
0	e^{-x}

Consulting the table above, we find that

$$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C.$$

- For the third integral, we set $u = \sqrt{x}$. This gives $x = u^2$ so that $dx = 2u\,du$ and

  $$\int e^{\sqrt{x}}\,dx = 2\int ue^{u}\,du.$$

  Using our computation from the previous problem, we conclude that

  $$\int e^{\sqrt{x}}\,dx = 2ue^{u} - 2e^{u} + C = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

- To compute the last integral, we integrate by parts using the choices

  $$\boxed{u = \log x, \quad dv = x^{-2}\,dx} \quad \Longrightarrow \quad \boxed{du = x^{-1}\,dx, \quad v = -x^{-1}.}$$

  Applying the formula for integration by parts, we find that

  $$\begin{aligned}\int x^{-2}\log x\,dx &= \int u\,dv = uv - \int v\,du \\ &= -x^{-1}\log x + \int x^{-2}\,dx = -x^{-1}\log x - x^{-1} + C.\end{aligned}$$