MA121 Tutorial Problems #3
Solutions

1. *Evaluate each of the following limits:*

$$\lim_{x\to+\infty} \frac{6x^2-5}{2-3x^2}, \qquad \lim_{x\to-\infty} \frac{6x^3-5x^2+2}{1-3x+x^4}.$$

- To find the limit of a rational function as $x \to \pm\infty$, one divides both the numerator and the denominator by the highest power of x in the denominator. In this case,

$$\lim_{x\to+\infty} \frac{6x^2-5}{2-3x^2} = \lim_{x\to+\infty} \frac{6-5/x^2}{2/x^2-3} = \frac{6-0}{0-3} = -2$$

and a similar computation gives

$$\lim_{x\to-\infty} \frac{6x^3-5x^2+2}{1-3x+x^4} = \lim_{x\to-\infty} \frac{6/x-5/x^2+2/x^4}{1/x^4-3/x^3+1} = \frac{0-0+0}{0-0+1} = 0.$$

2. *Find the minimum value of* $f(x) = (2x^2-5x+2)^3$ *over the closed interval* $[0,1]$.

- Since f is continuous on a closed interval, it suffices to check the endpoints, the points at which f' does not exist and the points at which f' is equal to zero. In this case,

$$\begin{aligned} f'(x) &= 3(2x^2-5x+2)^2 \cdot (2x^2-5x+2)' \\ &= 3(2x^2-5x+2)^2 \cdot (4x-5) \end{aligned}$$

is zero when $x = 5/4$ and also when the quadratic factor is zero, namely when

$$x = \frac{5 \pm \sqrt{25-4\cdot2\cdot2}}{2\cdot2} = \frac{5\pm3}{4} \quad \Longrightarrow \quad x=2, \quad x=1/2.$$

Since $x = 5/4$ and $x = 2$ do not lie in the given closed interval, this means that

$$x=0, \qquad x=1, \qquad x=1/2$$

are the only points at which the minimum value may occur. Once we now compute

$$f(0)=8, \qquad f(1)=-1, \qquad f(1/2)=0,$$

we may finally conclude that the minimum value is $f(1) = -1$.

3. *Show that* $\log x \le x - 1$ *for all* $x > 0$.

- Letting $f(x) = \log x - x + 1$ for convenience, one easily finds that
$$f'(x) = \frac{1}{x} - 1 = \frac{1-x}{x}.$$
Thus, $f'(x)$ is positive if and only if $1 - x > 0$, hence if and only if $x < 1$. This shows that f is increasing when $x < 1$ and also decreasing when $x > 1$, so
$$\max f(x) = f(1) = \log 1 - 1 + 1 = 0 \quad \Longrightarrow \quad f(x) \le \max f(x) = 0.$$

4. *Compute each of the following limits:*
$$L_1 = \lim_{x\to\infty} \frac{\log(x^2+1)}{x}, \qquad L_2 = \lim_{x\to 1} \frac{x^3 + x^2 - 5x + 3}{x^3 - x^2 - x + 1}.$$

- Since the first limit gives ∞/∞, we may apply L'Hôpital's rule to get
$$L_1 = \lim_{x\to\infty} \frac{\log(x^2+1)}{x} = \lim_{x\to\infty} \frac{(x^2+1)^{-1}\cdot(x^2+1)'}{1} = \lim_{x\to\infty} \frac{2x}{x^2+1}.$$
This is still an ∞/∞ limit, and another application of L'Hôpital's rule gives
$$L_1 = \lim_{x\to\infty} \frac{2x}{x^2+1} = \lim_{x\to\infty} \frac{2}{2x} = 0.$$

- The second limit gives $0/0$, so L'Hôpital's rule is applicable and we find
$$L_2 = \lim_{x\to 1} \frac{x^3 + x^2 - 5x + 3}{x^3 - x^2 - x + 1} = \lim_{x\to 1} \frac{3x^2 + 2x - 5}{3x^2 - 2x - 1}.$$
The last limit gives $0/0$ as well, so we may apply L'Hôpital's rule once again to get
$$L_2 = \lim_{x\to 1} \frac{3x^2 + 2x - 5}{3x^2 - 2x - 1} = \lim_{x\to 1} \frac{6x+2}{6x-2} = \frac{6+2}{6-2} = 2.$$

5. *Suppose that* $x > y > 0$. *Using the mean value theorem or otherwise, show that*
$$1 - \frac{y}{x} < \log x - \log y < \frac{x}{y} - 1.$$

- Letting $f(x) = \log x$ for convenience, we use the mean value theorem to find that
$$f'(c) = \frac{f(x) - f(y)}{x - y} \quad \Longrightarrow \quad \frac{1}{c} = \frac{\log x - \log y}{x - y}$$
for some $y < c < x$. Inverting these positive numbers reverses the inequality, hence
$$\frac{1}{x} < \frac{1}{c} < \frac{1}{y} \quad \Longrightarrow \quad \frac{1}{x} < \frac{\log x - \log y}{x - y} < \frac{1}{y}$$
$$\Longrightarrow \quad 1 - \frac{y}{x} < \log x - \log y < \frac{x}{y} - 1.$$