MA121 Tutorial Problems #2
Solutions

1. *Show that there exists some* $0 < x < 1$ *such that* $4x^3 + 3x = 2x^2 + 2$.

- Let $f(x) = 4x^3 + 3x - 2x^2 - 2$ for all $x \in [0, 1]$. Being a polynomial, f is continuous on the closed interval $[0, 1]$ and we also have

$$f(0) = -2 < 0, \qquad f(1) = 4 + 3 - 2 - 2 = 3 > 0.$$

In view of Bolzano's theorem, this means that $f(x) = 0$ for some $x \in (0, 1)$.

2. *Evaluate each of the following limits:*

$$\lim_{x\to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x + 1}, \qquad \lim_{x\to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x - 1}.$$

- When it comes to the first limit, one easily finds that

$$\lim_{x\to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x + 1} = \frac{6 - 5 - 3 + 2}{1 + 1} = \frac{0}{2} = 0$$

because rational functions are continuous at all points at which they are defined.

- When it comes to the second limit, division of polynomials gives

$$\lim_{x\to 1} \frac{6x^3 - 5x^2 - 3x + 2}{x - 1} = \lim_{x\to 1} (6x^2 + x - 2) = 6 + 1 - 2 = 5$$

because $x \neq 1$ and since polynomial functions are known to be continuous.

3. *Suppose that* f *is continuous on* $[0, 1]$ *and that* $0 < f(x) < 1$ *for all* $x \in [0, 1]$. *Show that there exists some* $0 < c < 1$ *such that* $f(c) = c$.

- Let $g(x) = f(x) - x$ for all $x \in [0, 1]$. Being the difference of two continuous functions, g is then continuous on the closed interval $[0, 1]$. Once we now note that

$$g(0) = f(0) > 0, \qquad g(1) = f(1) - 1 < 0,$$

we may use Bolzano's theorem to conclude that $g(c) = 0$ for some $c \in (0, 1)$. This also implies that $f(c) = c$ for some $0 < c < 1$, as needed.

4. *Let* f *be the function defined by*

$$f(x) = \left\{ \begin{array}{cc} 3x & \text{if } x \leq 1 \\ 4x - 1 & \text{if } x > 1 \end{array} \right\}.$$

Show that f *is continuous at all points.*

- Since f agrees with a polynomial on the open interval $(-\infty, 1)$ and polynomials are known to be continuous, it is clear that f is continuous on $(-\infty, 1)$. Using the exact same argument, we find that f is continuous on the open interval $(1, +\infty)$ as well.

- To check continuity at the remaining point $y = 1$, let us first note that

$$|f(x) - f(1)| = |f(x) - 3| = \left\{ \begin{array}{ll} 3|x-1| & \text{if } x \leq 1 \\ 4|x-1| & \text{if } x > 1 \end{array} \right\}.$$

Given any $\varepsilon > 0$, we can then set $\delta = \varepsilon/4$ to find that

$$|x - 1| < \delta \quad \Longrightarrow \quad |f(x) - f(1)| \leq 4|x - 1| < 4\delta = \varepsilon.$$

This establishes continuity at $y = 1$ as well, so f is continuous at all points.

5. *Determine the values of x for which $x^3 < 9x$.*

- We need to determine the values of x for which

$$f(x) = x^3 - 9x = x(x^2 - 9) = x(x - 3)(x + 3)$$

is negative. By the table below, this is true when either $x < -3$ or else $0 < x < 3$.

x		-3		0		3	
x	$-$		$-$		$+$		$+$
$x-3$	$-$		$-$		$-$		$+$
$x+3$	$-$		$+$		$+$		$+$
$f(x)$	$-$		$+$		$-$		$+$

6. *Show that the function f defined by*

$$f(x) = \left\{ \begin{array}{ll} 2x & \textit{if } x \leq 1 \\ x + 2 & \textit{if } x > 1 \end{array} \right\}$$

is not continuous at $y = 1$.

- We will show that the ε-δ definition of continuity fails when $\varepsilon = 1$. Suppose it does not fail. Since $f(1) = 2$, there must then exist some $\delta > 0$ such that

$$|x - 1| < \delta \quad \Longrightarrow \quad |f(x) - 2| < 1. \tag{$*$}$$

Let us now examine the last equation for the choice $x = 1 + \frac{\delta}{2}$. On one hand, we have

$$|x - 1| = \frac{\delta}{2} < \delta,$$

so the assumption in equation $(*)$ holds. On the other hand, we also have

$$|f(x) - 2| = |x + 2 - 2| = 1 + \frac{\delta}{2} > 1$$

because $x = 1 + \frac{\delta}{2} > 1$ here. This actually violates the conclusion in equation $(*)$.