MA121 Tutorial Problems #8

1. Classify the critical points of $f(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2$.

2. Classify the critical points of $f(x,y) = 3x^2 - 3y^2 + 8xy + 10x - 20y + 30$.

3. Compute each of the following integrals:

$$\int_0^1 \int_x^1 \cos(x/y)\, dy\, dx, \qquad \int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy, \qquad \int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\, dy\, dx.$$

- These are all practice problems, not a homework assignment.
- In case you get stuck, some hints are provided on the other page of this sheet.

Hints and comments

1. To find the critical points, you need to solve the equations

$$0 = f_x(x,y) = 4x(x^2 + 2y^2 - 1), \qquad 0 = f_y(x,y) = 4y(2x^2 + 1).$$

Note that $y = 0$ by the second equation and then simplify; the critical points are

$$(0,0), \qquad (1,0), \qquad (-1,0).$$

As it turns out, the last two are local minima, while the origin is a saddle point.

2. To find the critical points, you need to solve the equations

$$0 = f_x(x,y) = 6x + 8y + 10, \qquad 0 = f_y(x,y) = -6y + 8x - 20.$$

As it turns out, the only critical point is $(1,-2)$ and this is a saddle point.

3. For the first integral, switching the order of integration gives

$$\int_0^1 \int_x^1 \cos(x/y)\, dy\, dx = \int_0^1 \int_0^y \cos(x/y)\, dx\, dy$$

and one can easily compute the inner integral, namely

$$\int_0^y \cos(x/y)\, dx = \Big[y \sin(x/y) \Big]_{x=0}^{x=y} = y \sin 1.$$

- For the second integral, switching the order of integration gives

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx = \int_0^3 \frac{xe^{x^2}}{3}\, dx = \left[\frac{e^{x^2}}{6} \right]_0^3 = \frac{e^9 - 1}{6}.$$

- For the third integral, switching the order of integration gives

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\; dy\, dx = \int_0^2 \int_0^{3y} x\sqrt{y^3+1}\; dx\, dy = \int_0^2 \frac{9y^2\sqrt{y^3+1}}{2}\, dy$$

and one can now use the substitution $u = y^3 + 1$ to conclude that

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\, dy\, dx = \int_1^9 \frac{3\sqrt{u}}{2}\, du = \Big[u^{3/2} \Big]_1^9 = 26.$$