MA121 Tutorial Problems #6

1. Find the radius of convergence for each of the following power series:

$$\sum_{n=0}^{\infty} \frac{nx^n}{3^n}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n\, x^n}{2n+1}, \qquad \sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} \cdot x^n.$$

2. Although a power series may be differentiated term by term, this is not really the case for an arbitrary series. In fact, an infinite sum of continuous/differentiable functions does not even have to be continuous/differentiable itself. To see this, let

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \qquad f_n(x) = \frac{x^2}{(1+x^2)^n}$$

and check that f is not continuous at $x = 0$, even though each f_n is.

3. Consider the function f defined by the power series

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(a) Show that f is defined for all $x \in \mathbb{R}$ and that we also have

$$f'(x) = f(x), \qquad f(0) = 1.$$

(b) Use part (a) to show that $f(x)f(-x) = 1$ and that $f(x) > 0$ for all $x \in \mathbb{R}$.

- These are all practice problems, not a homework assignment.
- However, part of your next homework assignment will be based on these problems.
- In case you get stuck, some hints are provided on the other page of this sheet.

Hints and comments

1a. One always determines the radius of convergence using the ratio test. In this case,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{|x|^{n+1}}{|x|^n} \cdot \frac{3^n}{3^{n+1}} = \frac{|x|}{3}$$

so the series converges when $|x|/3 < 1$ and diverges when $|x|/3 > 1$. That is, the series converges when $|x| < 3$ and diverges when $|x| > 3$. This also means that $R = 3$.

1b. Following the argument of part (a), you should find that $R = 1$.

1c. Following the argument of part (a), you should find that $R = 4$. More precisely,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} \cdot \frac{(2n)!}{(2n+2)!} \cdot \frac{x^{n+1}}{x^n} = \frac{(n+1)^2 \cdot x}{(2n+1)(2n+2)}$$

and this implies that

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n^2+2n+1)\,|x|}{4n^2+6n+2} = \frac{|x|}{4}.$$

2. Each f_n is a rational function which is defined at all points, so each f_n is continuous at all points. To show that f is not continuous, use the formula for a geometric series to show that $f(x) = 1 + x^2$ for all $x \neq 0$, note that $f(0) = 0$ and conclude that

$$\lim_{x\to 0} f(x) \neq f(0).$$

3a. To check that the series converges for all x, you need to use the ratio test. To check that it satisfies $f'(x) = f(x)$, you can differentiate the series term by term to get

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \Longrightarrow \quad f'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \ldots$$

and then you can simplify to conclude that $f'(x) = f(x)$.

3b. First of all, check that $g(x) = f(x)f(-x)$ is constant by showing that $g'(x) = 0$. To prove the inequality, note that f is never zero since $f(x)f(-x) = 1$ at all points. Why does this imply f is either positive at all points or else negative at all points?