MA121 Tutorial Problems #5

1. This is the correct version of an example I messed up last week. Let $x > 0$ be given and define a sequence $\{a_n\}$ by picking any number $a_1 \geq \sqrt{x}$ and by setting

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) \quad \text{for each } n \geq 1.$$

Show that $a_n \geq \sqrt{x}$ for all n, that $\{a_n\}$ is decreasing and that $\lim_{n\to\infty} a_n = \sqrt{x}$.

2. Compute each of the following sums:

$$\sum_{n=0}^{\infty} \frac{2^n}{7^n}\,, \qquad \sum_{n=0}^{\infty} \frac{3^{n+2}}{2^{2n+1}}\,, \qquad \sum_{n=1}^{\infty} \frac{5^{n+1}}{2^{3n}}\,.$$

3. Show that the sequence defined by $a_n = n\sin(1/n)$ is convergent.

4. Test each of the following series for convergence:

$$\sum_{n=1}^{\infty} n\sin(1/n)\,, \qquad \sum_{n=1}^{\infty} \frac{1}{n+e^n}\,, \qquad \sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}\,, \qquad \sum_{n=1}^{\infty} \frac{\log n}{n}\,.$$

- These are all practice problems, not a homework assignment.
- However, part of your next homework assignment will be based on these problems.
- In case you get stuck, some hints are provided on the other page of this sheet.

Hints and comments

1. To show that $a_n \geq \sqrt{x}$ for all n, note that

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) = \frac{1}{2}\left(\sqrt{a_n} - \sqrt{\frac{x}{a_n}}\right)^2 + \sqrt{x}.$$

To show that the sequence is decreasing, note that

$$a_n^2 \geq x \quad \Longrightarrow \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{x}{a_n}\right) \leq \frac{1}{2}(a_n + a_n) = a_n.$$

2. The answers are 7/5, 18 and 25/3, respectively.

3. Write $a_n = \frac{\sin(1/n)}{1/n}$ and then use L'Hôpital's rule; the limit is equal to 1.

4a. Since the nth term fails to approach zero, the series diverges.

4b. Use the comparison test. This series is smaller than the geometric series $\sum \frac{1}{e^n}$.

4c. Note that $e^{1/n} \leq e$ and then use comparison.

4d. Note that $\sum_{n=1}^{\infty} \frac{\log n}{n} = \sum_{n=2}^{\infty} \frac{\log n}{n} \geq \sum_{n=2}^{\infty} \frac{\log 2}{n}$ and then use comparison.