MA121, Sample Exam #3
Solutions

1. *Let A, B be nonempty subsets of $\mathbb{R}$ such that $\inf A < \inf B$. Show that there exists an element $a \in A$ which is a lower bound of B.*

- Since $\inf B$ is bigger than the greatest lower bound of A, we see that $\inf B$ cannot be a lower bound of A. This means that some element $a \in A$ is such that $a < \inf B$. Using the fact that $\inf B$ is a lower bound of B, we conclude that $a < \inf B \leq b$ for all $b \in B$. This also means that a itself is a lower bound of B.

2. *Let f be the function defined by*

$$f(x) = \left\{ \begin{array}{ll} 3x+1 & \text{if } x \in \mathbb{Q} \\ 6-2x & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Show that f is continuous at $y = 1$.

- To prove that f is continuous at $y = 1$, let us first note that

$$|f(x) - f(1)| = |f(x) - 4| = \left\{ \begin{array}{ll} 3|x-1| & \text{if } x \in \mathbb{Q} \\ 2|1-x| & \text{if } x \notin \mathbb{Q} \end{array} \right\}.$$

Now, let $\varepsilon > 0$ be given and set $\delta = \varepsilon/3$. Then $\delta > 0$ and we have

$$|x-1| < \delta \quad \Longrightarrow \quad |f(x) - f(1)| \leq 3|x-1| < 3\delta = \varepsilon.$$

3. *Show that $2e \cdot x^2 \log x \geq -1$ for all $x > 0$. Here, e is the usual constant $e \approx 2.718$.*

- Letting $f(x) = x^2 \log x$ for convenience, one easily finds that

$$f'(x) = 2x \log x + x^2 \cdot x^{-1} = 2x \log x + x = x(2 \log x + 1).$$

Since $x > 0$ by assumption, the given function is then increasing if and only if

$$2 \log x + 1 > 0 \quad \Longleftrightarrow \quad \log x > -1/2 \quad \Longleftrightarrow \quad x > e^{-1/2}.$$

This means f is decreasing when $0 < x < e^{-1/2}$ and increasing when $x > e^{-1/2}$, so

$$f(x) \geq f(e^{-1/2}) = e^{-1} \log e^{-1/2} = -\frac{1}{2e} \quad \Longrightarrow \quad 2e \cdot f(x) \geq -1.$$

4. *Compute each of the following integrals:*

$$\int \frac{4x^2 - 5x + 2}{x^3 - x^2}\, dx, \qquad \int \sin^3 x\, dx.$$

- To compute the first integral, we factor the denominator and we write

$$\frac{4x^2-5x+2}{x^3-x^2} = \frac{4x^2-5x+2}{x^2(x-1)} = \frac{Ax+B}{x^2} + \frac{C}{x-1}$$

for some constants A, B, C that need to be determined. Clearing denominators gives

$$4x^2-5x+2 = (Ax+B)(x-1) + Cx^2$$

and we can now look at some suitable choices of x to find that

$$x = 0, 1, 2 \quad \Longrightarrow \quad 2 = -B, \quad 1 = C, \quad 8 = 2A + B + 4C.$$

Since the last equation gives $2A = 8 - B - 4C = 6$, we get

$$\frac{4x^2-5x+2}{x^3-x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{3}{x} - \frac{2}{x^2} + \frac{1}{x-1}\,.$$

Once we now integrate this equation term by term, we may finally conclude that

$$\int \frac{4x^2-5x+2}{x^3-x^2}\,dx = 3\log|x| + 2x^{-1} + \log|x-1| + C.$$

- To compute the second integral, it is convenient to write it in the form

$$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx = \int \sin x\,dx - \int \cos^2 x \sin x\,dx.$$

Using the substitution $u = \cos x$, we then get $du = -\sin x\,dx$, hence also

$$\int \sin^3 x\,dx = -\cos x + \int u^2\,du = -\cos x + \frac{u^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C.$$

5. *Using the mean value theorem, or otherwise, show that*

$$(b-a)e^a < e^b - e^a < (b-a)e^b \quad \textit{whenever } a < b.$$

- Since $f(x) = e^x$ is differentiable on $[a, b]$, the mean value theorem applies to give

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \Longrightarrow \quad e^c = \frac{e^b - e^a}{b-a}$$

for some $c \in (a, b)$. We now use the fact that f is strictly increasing to find that

$$a < c < b \quad \Longrightarrow \quad e^a < e^c < e^b \quad \Longrightarrow \quad e^a < \frac{e^b-e^a}{b-a} < e^b.$$

Multiplying by the positive quantity $b - a$, we thus obtain the desired inequality.

6. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 5^n}, \qquad \sum_{n=1}^{\infty} \sin(1/n^2).$$

- To test the first series for convergence, we use the comparison test. Since

$$\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 5^n} \leq \sum_{n=1}^{\infty} \frac{2^n + 4^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n,$$

the first series is smaller than the sum of two convergent series, so it converges.

- For the second series, we use the limit comparison test with

$$a_n = \sin(1/n^2) = \sin n^{-2}, \qquad b_n = 1/n^2 = n^{-2}.$$

To show that the limit comparison test is applicable here, we need to show that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin n^{-2}}{n^{-2}}$$

is equal to 1. Noting that this is a 0/0 limit, we may use L'Hôpital's rule to get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\cos n^{-2} \cdot (n^{-2})'}{(n^{-2})'} = \lim_{n\to\infty} \cos n^{-2} = \cos 0 = 1.$$

Since $\sum_{n=1}^{\infty} b_n$ is a convergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must then converge as well.

7. *Suppose f, g are integrable on $[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$. Show that*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

- Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Starting with the inequality

$$f(x) \leq g(x) \quad \text{for all } \ x \in [x_k, x_{k+1}],$$

we take the infimum of both sides to get

$$\inf_{[x_k, x_{k+1}]} f(x) \leq \inf_{[x_k, x_{k+1}]} g(x).$$

Multiplying by the positive quantity $x_{k+1} - x_k$ and then adding, we conclude that

$$\sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} f(x) \cdot (x_{k+1} - x_k) \leq \sum_{k=0}^{n-1} \inf_{[x_k, x_{k+1}]} g(x) \cdot (x_{k+1} - x_k).$$

Since the last inequality holds for all partitions P by above, we must thus have

$$S^-(f, P) \leq S^-(g, P)$$

for all partitions P. Taking the supremum of both sides, we finally deduce that

$$\int_a^b f(x)\,dx = \sup_P \{S^-(f, P)\} \ \leq \ \sup_P \{S^-(g, P)\} = \int_a^b g(x)\,dx.$$

8. *Letting* $f(x,y) = \log(x^2+y^2)$*, find the rate at which* f *is changing at the point* $(2,3)$ *in the direction of the vector* $\mathbf{v} = \langle 3,4 \rangle$*.*

- To find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$, we need to divide $\mathbf{v}$ by its length, namely

$$||\mathbf{v}|| = \sqrt{3^2+4^2} = 5 \quad \Longrightarrow \quad \mathbf{u} = \frac{1}{5}\mathbf{v} = \langle 3/5, 4/5 \rangle .$$

The desired rate of change is given by the directional derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$. Since

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle \quad \Longrightarrow \quad \nabla f(2,3) = \langle 4/13, 6/13 \rangle ,$$

we may thus conclude that the desired rate of change is

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{4}{13} \cdot \frac{3}{5} + \frac{6}{13} \cdot \frac{4}{5} = \frac{36}{65} .$$

9. *Classify the critical points of the function defined by* $f(x,y) = x^2 + 2y^2 - x^2y$*.*

- To find the critical points, we need to solve the equations

$$0 = f_x(x,y) = 2x - 2xy = 2x(1-y),$$
$$0 = f_y(x,y) = 4y - x^2.$$

If $x = 0$, then $y = 0$ by the second equation. Otherwise, $y = 1$ by the first equation, so

$$x^2 = 4y = 4 \quad \Longrightarrow \quad x = \pm 2$$

by the second equation. In particular, the only critical points are $(0,0)$ and $(\pm 2, 1)$.

- To classify the critical points, we compute the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2-2y & -2x \\ -2x & 4 \end{bmatrix} .$$

When it comes to the critical points $(\pm 2, 1)$, this gives

$$H = \begin{bmatrix} 0 & \mp 4 \\ \mp 4 & 4 \end{bmatrix} \quad \Longrightarrow \quad \det H = -16 < 0$$

so each of those is a saddle point. When it comes to the critical point $(0,0)$, we have

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad \Longrightarrow \quad \det H = 8 > 0$$

so the fact that $f_{xx} = 2 > 0$ makes the origin a local minimum.

10. *Compute the double integrals*

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx, \qquad \int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy, \qquad \int_0^2 \int_{x^2}^4 x e^{y^2}\, dy\, dx.$$

- To compute the first integral, we switch the order of integration to get

$$\begin{aligned}\int_0^\pi \int_x^\pi \frac{\sin y}{y}\, dy\, dx &= \int_0^\pi \int_0^y \frac{\sin y}{y}\, dx\, dy = \int_0^\pi \sin y\, dy \\ &= \Big[-\cos y\Big]_0^\pi = -\cos\pi + \cos 0 = 2.\end{aligned}$$

- When it comes to the second integral, switching the order of integration gives

$$\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy = \int_0^1 \int_0^x x^2 e^{xy}\, dy\, dx.$$

We temporarily focus on the inner integral, which is given by

$$\int_0^x x^2 e^{xy}\, dy = x^2 \int_0^x e^{xy}\, dy = x^2 \left[\frac{e^{xy}}{x}\right]_{y=0}^{y=x} = x(e^{x^2} - 1).$$

Once we now combine the last two equations, we arrive at

$$\int_0^1 \int_y^1 x^2 e^{xy}\, dx\, dy = \int_0^1 (xe^{x^2} - x)\, dx = \left[\frac{e^{x^2}}{2} - \frac{x^2}{2}\right]_0^1 = \frac{e-2}{2}.$$

- To compute the last integral, we switch the order of integration to get

$$\int_0^2 \int_{x^2}^4 x e^{y^2}\, dy\, dx = \int_0^4 \int_0^{\sqrt{y}} x e^{y^2}\, dx\, dy.$$

We temporarily focus on the inner integral, which is given by

$$\int_0^{\sqrt{y}} x e^{y^2}\, dx = \left[\frac{x^2 e^{y^2}}{2}\right]_{x=0}^{x=\sqrt{y}} = \frac{ye^{y^2}}{2}.$$

Once we now combine the last two equations, we arrive at

$$\int_0^2 \int_{x^2}^4 x e^{y^2}\, dy\, dx = \int_0^4 \frac{ye^{y^2}}{2}\, dy = \left[\frac{e^{y^2}}{4}\right]_0^4 = \frac{e^{16}-1}{4}.$$