MA121, Homework #8
Solutions

1. *Compute the partial derivatives f_x and f_y in the case that $f(x,y) = \arctan(x/y)$.*

- According to the chain rule, we have

$$f_x = \frac{1}{1+(x/y)^2} \cdot (x/y)_x = \frac{y^2}{x^2+y^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2},$$
$$f_y = \frac{1}{1+(x/y)^2} \cdot (x/y)_y = \frac{y^2}{x^2+y^2} \cdot \frac{-x}{y^2} = -\frac{x}{x^2+y^2}.$$

2. *Compute the partial derivatives f_x and f_y in the case that $f(x,y) = x^2 \sin(xy)$.*

- To compute f_x, we combine the product rule with the chain rule to get

$$f_x = 2x\sin(xy) + x^2\cos(xy) \cdot (xy)_x = 2x\sin(xy) + x^2 y\cos(xy).$$

The computation of f_y is much easier, namely

$$f_y = x^2\cos(xy) \cdot (xy)_y = x^3\cos(xy).$$

3. *Letting $f(x,y) = \log(x^2+y^2)$, find the rate at which f is changing at the point $(1,2)$ in the direction of the vector $\mathbf{v} = \langle 2,1 \rangle$.*

- To find a unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$, let us first divide $\mathbf{v}$ by its length:

$$||\mathbf{v}|| = \sqrt{2^2+1^2} = \sqrt{5} \quad \Longrightarrow \quad \mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{v} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle.$$

The desired rate of change is given by the directional derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$. Since

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle \quad \Longrightarrow \quad \nabla f(1,2) = \langle 2/5, 4/5 \rangle,$$

the desired rate of change is then

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{2}{5} \cdot \frac{2}{\sqrt{5}} + \frac{4}{5} \cdot \frac{1}{\sqrt{5}} = \frac{8}{5\sqrt{5}} = \frac{8\sqrt{5}}{25}.$$

4. *Suppose that $z = e^x \cos y$, where $x = st$ and $y = \log(s^2+t^2)$. Compute z_s and z_t.*

- According to the chain rule, we have

$$z_s = z_x x_s + z_y y_s = te^x\cos y - \frac{2se^x \sin y}{s^2+t^2},$$
$$z_t = z_x x_t + z_y y_t = se^x\cos y - \frac{2te^x \sin y}{s^2+t^2}.$$

5. *Suppose that* $f = f(u, v, w)$, *where* $u = x - y$, $v = y - z$ *and* $w = z - x$. *Assuming that all partial derivatives exist, show that* $f_x + f_y + f_z = 0$.

- First of all, we use three applications of the chain rule to get

$$\begin{aligned} f_x &= f_u u_x + f_v v_x + f_w w_x = f_u - f_w, \\ f_y &= f_u u_y + f_v v_y + f_w w_y = -f_u + f_v, \\ f_z &= f_u u_z + f_v v_z + f_w w_z = -f_v + f_w. \end{aligned}$$

  Once we now add these three equations, we find that $f_x + f_y + f_z = 0$, indeed.