MA121, Homework #7
Solutions

1. *Find the area of the region that lies between the graphs of $f(x) = 4x$ and $g(x) = 2x^2$.*

- First of all, let us note that the two graphs intersect when

$$4x = 2x^2 \quad \Longrightarrow \quad 2x(2-x) = 0 \quad \Longrightarrow \quad x = 0, 2.$$

As one can see from a rough sketch, the graph of f lies above the graph of g between these two points, so the desired area is

$$\int_0^2 [f(x) - g(x)]\, dx = \int_0^2 \left[4x - 2x^2\right] dx = \left[2x^2 - \frac{2x^3}{3}\right]_0^2 = \frac{8}{3}.$$

2. *Let R be the region between the graph of $f(x) = \cos x$ and the x-axis over $[0, \pi]$. Find the volume of the solid obtained upon rotation of R around the x-axis.*

- Rotating R around the x-axis, one obtains a solid whose volume is given by

$$\pi \int_0^\pi \cos^2 x\, dx = \pi \int_0^\pi \frac{1 + \cos(2x)}{2}\, dx = \frac{\pi}{2}\left[x + \frac{\sin(2x)}{2}\right]_0^\pi = \frac{\pi^2}{2}.$$

3. *Let R be the region between the graph of $f(x) = e^x - 1$ and the x-axis over $[0, 1]$. Find the volume of the solid obtained upon rotation of R around the x-axis.*

- Rotating R around the x-axis, one obtains a solid whose volume is given by

$$\begin{aligned} \pi \int_0^1 (e^x - 1)^2\, dx &= \pi \int_0^1 (e^{2x} - 2e^x + 1)\, dx \\ &= \pi \left[\frac{e^{2x}}{2} - 2e^x + x\right]_0^1 = \frac{\pi(e^2 - 4e + 5)}{2}. \end{aligned}$$

4. *In each case, compute the limit or else show that it does not exist:*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + 2y^2}, \qquad \lim_{(x,y)\to(0,0)} \frac{x^2 y}{3x^2 + y^2}, \qquad \lim_{(x,y)\to(2,1)} \frac{2x^2 - xy - 6y^2}{x - 2y}.$$

- For the first limit, we use polar coordinates to write the given fraction as

$$f(x,y) = \frac{xy}{x^2 + 2y^2} = \frac{r^2 \cos\theta \sin\theta}{r^2 \cos^2\theta + 2r^2 \sin^2\theta} = \frac{\cos\theta \sin\theta}{1 + \sin^2\theta}.$$

Since this expression depends only on θ, we see that the first limit does not exist.

- For the second limit, we use polar coordinates to similarly write

$$f(x,y) = \frac{x^2y}{3x^2+y^2} = \frac{r^3\cos^2\theta\sin\theta}{3r^2\cos^2\theta + r^2\sin^2\theta} = \frac{r\cos^2\theta\sin\theta}{2\cos^2\theta+1}.$$

  Since (x,y) approaches the origin, we have $r = \sqrt{x^2+y^2} \to 0$, so the given function must approach zero as well. More precisely, we have

$$0 \leq |f(x,y)| \leq |r\cos^2\theta\sin\theta| \leq r$$

  and the fact that $r \to 0$ implies that $f(x,y) \to 0$ because of the Squeeze Law.

- Using division of polynomials to compute the last limit, one finds that

$$\lim_{(x,y)\to(2,1)} \frac{2x^2-xy-6y^2}{x-2y} = \lim_{(x,y)\to(2,1)} (2x+3y) = 4+3 = 7$$

  because the linear function $g(x,y) = 2x+3y$ is known to be continuous.