MA121, Homework #6
Solutions

1. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}\,, \qquad \sum_{n=1}^{\infty} \frac{n}{2^n}\,, \qquad \sum_{n=1}^{\infty} \frac{n}{n^2+2^n}\,, \qquad \sum_{n=1}^{\infty} \left(1-\frac{1}{n^2}\right)^n.$$

- To test the first series for convergence, we use the limit comparison test with

$$a_n = \frac{1}{\sqrt{n+1}}\,, \qquad b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}\,.$$

Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 1.$$

Since the series $\sum_{n=1}^{\infty} b_n$ is a divergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must also diverge.

- To test the second series for convergence, we use the ratio test. In this case,

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2}$$

is strictly less than 1, so the second series converges by the ratio test.

- For the third series, we use the comparison test. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^2+2^n} \leq \sum_{n=1}^{\infty} \frac{n}{2^n}$$

and the rightmost series converges by above, the leftmost series converges as well.

- For the last series, finally, we use the nth term test. In this case, we have

$$\lim_{n\to\infty} \left(1-\frac{1}{n^2}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{n}\right)^n = e^1 e^{-1} = 1.$$

Since the nth term fails to approach zero, the last series must thus diverge.

2. *Test each of the following series for convergence:*

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}\,, \qquad \sum_{n=1}^{\infty} \frac{1}{n} \cdot \log\left(1 + \frac{1}{n}\right), \qquad \sum_{n=1}^{\infty} \frac{1}{n^n}\,, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\,.$$

- To test the first series for convergence, we use the ratio test. In this case,

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

  is strictly less than 1, so the first series converges by the ratio test.

- To test the second series for convergence, we use the limit comparison test with

$$a_n = \frac{1}{n} \cdot \log\left(1 + \frac{1}{n}\right), \qquad b_n = \frac{1}{n^2}\,.$$

  Note that the limit comparison test is, in fact, applicable here because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n \cdot \log\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} \log\left(1 + \frac{1}{n}\right)^n = \log e = 1.$$

  Since $\sum_{n=1}^{\infty} b_n$ is a convergent p-series, the series $\sum_{n=1}^{\infty} a_n$ must also converge.

- To test the third series for convergence, one can use the ratio test to get

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} = \frac{1}{e} \cdot 0 = 0;$$

  this implies convergence due to the ratio test. Alternatively, one can argue that

$$\sum_{n=1}^{\infty} \frac{1}{n^n} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^n} \leq 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$$

  and that this implies convergence due to the comparison test.

- When it comes to the last series, convergence follows easily by the alternating series test because $|a_n| = 1/n$ is positive and decreasing with

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0.$$